Various models for symmetric diffusions
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1 Introduction

The aim of this short survey is to describe various models of diffusion processes where explicit computations are at hand. Most of them arise from geometric models encountered in various settings, and are associated with different families of orthogonal polynomials. The aim is mainly to collect in a common place the various formulae spread out in many different notes.

We shall see that those difusion processes and their laws, that is diffusion semigroups and their generators, offer a rich interplay between probability, analysis, geometry and and algebra.

Basically, we shall consider Brownian motions on euclidean spaces, spheres, Lie groups such as $SO(n)$, $SU(n)$, on symmetric matrices, Hermitian matrices, together with various projections : spectral measures, invariance under sub groups (discrete and continuous), radial parts, etc.

In these notes, a special attention will be devoted to specific models linked with multivariate orthogonal polynomials, one the one hand because of some wonderful structure involved, and on the other one for their importance in many applications.

The diffusions that we are concerned with are symmetric diffusions. Their are described by 3 objects : an algebra of functions, a carré du champ perator Γ, and a measure, which in most cases (not always) will be a probability measure. This is what is called a Markov triple.

2 Plan of the course

1. Generalities on symmetric diffusion operators and semigroups
2. Classical models : Brownian motions in Euclidean, spherical and hyperbolic spaces, Sturm Liouville operators in dimension 1, Brownian motion on Lie groups, $SO(d)$ and $SU(d)$.
3. Models with polynomial eigenvectors.
4. Spectral measures
5. The hypergroup property and Gasper's theorem
6. Models on boundaries
7. The 2-d and 3-d examples : a word about invariant theory

3 Symmetric diffusion operators

Symmetric diffusion operators and their associated heat semigroups play a central rôle in the study of continuous Markov processes, but also in differential geometry and partial differential equations. The analysis of the associated heat or potential kernels have been considered from many points of view, short and long time asymptotics, upper and lower bounds, on and away from the diagonal, convergence to equilibrium, e.g. All these topics had been deeply investigated along the past half century, see [1, 5, ?] for example. Unfortunately, there are very few examples where computations are explicit.

We shall always consider diffusions with values in some open set in $\mathbb{R}^d$, or on a manifold (spheres or sets of matrices).

3.1 Stochastic differential equations

Formally, a diffusion process s is a continuous Markov process ξ_t with continuous path with values in a topological space X, such that, for any initial point $x \in X$, and any set of times $(t_1 \leq t_2 \leq \cdots \leq t_n)$, the law of ξ_{t_n} given $(\xi_0, \xi_{t_1}, \cdots, \xi_{t_{n-1}})$ is the law of ξ_{t_n} given $\xi_{t_{n-1}}$ and also the law of $\xi_{t_n - t_{n-1}}$ given ξ_0.

In probability, diffusion processes arise when one solves a stochastic differential equation

$$d\xi_t = \sigma(\xi_t)dB_t + b(\xi_t)dt, \ \xi_0 = x.$$

Here, to fix the ideas, $\xi_t \in \mathbb{R}^d$, B_t is a n dimensional Brownian motion, $x \mapsto \sigma(x)$ is a smooth field of $n \times d$ matrices, and $x \mapsto b(x)$ is a smooth fields of d dimensional vectors.

In order to describe the law of such objects, one is led to compute, for a large class of functions f,

$$P_t(f)(x) = \mathbb{E}_x(f(\xi_t)).$$

A simple application of Ito's formula provides the fact that

$$f(\xi_t) = f(x) + M_t^f + \int_0^t \mathrm{L}(f)\xi_s)ds,$$

where M_t^f is a local martingale and

$$\mathrm{L}(f) = \frac{1}{2}\sum_{ij}(\sigma\sigma^*)_{ij}(x)\partial^2_{ij}f + \sum_i b_i(x)\partial_i f.$$

L is a second order differential operator with no zero order term, moreover semi-elliptic (we shall come back to that later), that we call a diffusion operator.

In nice situations, M^f is a real martingale, and taking expectations in Ito's formula, one gets

$$P_t(f)(x) = f(x) + \int_0^t P_s \mathrm{L} f(x) ds.$$

Moreover, if the stochastic differential equation has a unique solution, one also gets

$$P_t \circ P_s = P_{t+s},$$

which is called the semigroup property.

We see then that $P_t f(x)$ is the solution of the heat equation

$$\partial_t P_t(f) = \mathrm{L} P_t(f) = P_t \mathrm{L}(f).$$

Formally, $P_t = \exp(t\mathrm{L})$, and the main question is to describe this operator P_t as much as we can from the description of L.

Even with the knowledge of L, it is not easy to obtain good expressions for the knowledge of P_t, (or in an equivalent way for the law of ξ_t. There are indeed very few cases where this law is known explicitely : Brownian motion Ornstein-Uhlenbeck processes, odd dimensional hyperbolic Brownian motion, some nilpotent Lie groups (however under an integral representation which is not really easy to handle), and perhaps a few other cases.

The case is a bit simpler when the process is reversible, or symmetric. This means that there exists a measure (not necesserily a probability measure) for which the operator L is symmetric : for any pair of good functions f, g, one has

$$\int f \mathrm{L} g d\mu = \int g \mathrm{L} f d\mu.$$

This is the case of all the above mentioned examples : for the standard Brownian motion, this measure is the Lebesgue measure.

In this situation, one may use the spectral decomposition of L, which is then a self-adjoint operator. To fix the ideas, assume that the spectrum is discrete : one may then find an orthonormal basis (f_n) of $\mathcal{L}^2(\mu)$ such that $\mathrm{L}(f_n) = -\lambda_n f_n$, for some sequence of non negative real numbers λ_n which are called the eigenvalues (the fact that the eigenvalues are non negative will be explained below).

Using this spectral decomposition, one may write $P_t(f_n) = e^{-\lambda_n t} f_n$, so that writing

$$p_t(x, y) = \sum_n e^{-\lambda_n t} f_n(x) f_n(y),$$

provided this series converges in some reasonable sense, for example as soon as $\sum_n e^{-2\lambda_n t} < \infty$, one has

$$P_t f(x) = \int f(y) p_t(x, y) \mu(dy).$$

(This is immediately checked when $f = f_n$ and extended on any $\mathcal{L}^2$ function f by linearity). Then, one sees that the law of ξ_t knowing that $\xi_0 = x$, is nothing else than $p_t(x, y)\mu(dy)$.

Unfortunately, this representation is not very useful. It is not even at all easy to see on it that $p_t(x, y) \geq 0$.

3.2 Generators and carré du champ

From now on, we start from the generator L, which is a second order differential operator with no 0-order term, moreover semi-elliptic.

To be more precise, we shall consider some open connected set $\Omega \subset \mathbb{R}^d$, with piecewise smooth boundary $\partial\Omega$ (say at least piecewise $\mathcal{C}^1$, and may be empty).

A diffusion operator L on Ω (or on a smooth manifold) is a linear second order differential operator with no 0-order terms

$$\mathrm{L}(f) = \sum_{ij} g^{ij}(x) \partial^2_{ij} f + \sum_i b^i(x) \partial_i(f), \tag{3.1}$$

such that at every point $x \in \Omega$, the symmetric matrix $(g^{ij}(x))$ is non negative (this is the meaning of semi-ellipticity) It is said to be elliptic when moreover this matrix $(g^{ij})(x)$ is everywhere non degenerate in Ω, that we shall assume in what follows for simplicity (it may, and will in general, be degenerate at the boundary $\partial\Omega$). We will also assume that the coefficients $g^{ij}(x)$ and $b^i(x)$ are smooth.

One introduces the carré du champ

$$\Gamma(f, g) = \frac{1}{2}(\mathrm{L}(fg) - f\mathrm{L}(g) - g\mathrm{L}(f)).$$

In the above representation,

$$\Gamma(f, g) = \sum_{ij} g^{ij}(x) \partial_i f \partial_j g.$$

It is a first order operator in f and g, while L is second order. If $f = (f_1, \cdots, f_k)$, and $\Phi : \mathbb{R}^k \mapsto \mathbb{R}$ is smooth,

$$\begin{cases} \Gamma(\Phi(f)), g) = \sum_i \partial_i \Phi(f) \Gamma(f, g), \\ \mathrm{L}(\Phi(f)) = \sum_i \partial_i \Phi(f) \mathrm{L}(f_i) + \sum_{ij} \partial^2_{ij} \Phi(f) \Gamma(f_i, f_j). \end{cases}$$

These formulas (known as the change of variable formulas) allow to determine the action of L and Γ on any functions $f(x)$ and $g(x)$ as soon as one knows $\mathrm{L}(x_i)$ and $\Gamma(x_i, x_j)$. Observe that

$$\mathrm{L}(x_i) = b^i, \ \Gamma(x_i, x_j) = g^{ij}.$$

From this and the change of variable formula, one recovers the usual form of the operator in a system of coordinates.

These change of variable formulas are some abstract way to say that L is a second order differential operator.

The non negativity of the matrix g at any point translates into the fact that $\Gamma(f, f) \geq 0$ for any smooth functions.

Observe that the semi-ellipticity translates into the fact that $\Gamma(f, f) \geq 0$.

Formally, those two properties

1. $\Gamma(f, f) \geq 0$
2. $\mathrm{L}(1) = 0$

caracterize the operators L such that $P_t = \exp(tL)$ is a Markov operator, that is

$$f \geq 0 \implies P_t(f) \geq 0 \text{ and } P_t(\mathbf{1}) = \mathbf{1}.$$

If one wants to compare with the finite set setting, one wants to describe finite matrices $L = (L_{ij})$ such that $\exp(tL)$ satisfy the same property. In this case, one sees a function on the points $\{1, \cdots, n\}$ as a vector $(f_1, \cdots, f_n)$ (with $f_i = f(i)$), and $\mathrm{L}(f)_i = \sum_j L_{ij} f_j$.

It is quite an elementary exercise to see that the necessary and sufficient conditions for that is that

1. $\forall i \neq j$, $L_{ij} \geq 0$
2. $\forall i$, $\sum_j L_{ij} = 0$.

It is quite immediate that in this case these condition are equivalent to the previous ones.

Moreover, if P_t has to be a Markov operator, Cauchy-Schwartz inequality will imply $P_t(f^2) \geq (P_t f)^2$, which it $t = 0$ provides $\Gamma(f, f) \geq 0$.

The reverse implication, however, is much more technical in general and requires further hypotheses on L (see [1] for precise statements).

Formally, this is the translation of the positivity preserving property of $P_t = \exp(t\mathrm{L})$.

In general, the operator P_t acts on any bounded measurable function (this is an expectation with respect to a probability measure), whereas the operators L and Γ only act on good smooth functions. the relation between L and P_t is that P_t solves the heat equation

$$\partial_t P_t f = P_t \mathrm{L} f = \mathrm{L} P_t f,\ P_0(f) = f.$$

However, it is not clear how the description of L and some set of good functions allows to assert a unique solution to this equation. This is the question of the determination of a core, that we shall come to in the next section in the context of symmetric diffusion.

For the moment, we shall assume that those functions on which L and Γ act is some algebra $\mathcal{A}$ of functions. In $\mathbb{R}^d$ or some open set $\Omega \subset \mathbb{R}^d$, one may think of $\mathcal{A}$ to be the set of smooth (that is $\mathcal{C}^\infty$) functions which are compactly supported in Ω (we shall be more precise below).

3.3 Complex variables

Quite often, the coefficients g^{ij} and b^i are analytic (they will be even polynomials in most examples). Then, one may pair two variables (or any function) (say x and y), and set $z = x + iy$, $\bar{z} = x - iy$. Then, one may compute $\Gamma(z,z)$, $\Gamma(\bar{z},\bar{z})$, $\Gamma(z,\bar{z})$, $\mathrm{L}(z$ and $\mathrm{L}(\bar{z})$, by linearity and bilinearity. as such

$$\begin{cases} \mathrm{L}(z) = \mathrm{L}(x) + i\mathrm{L}(y),\ \mathrm{L}(\bar{z}) = \mathrm{L}(x) - i\mathrm{L}(y) = \overline{\mathrm{L}(z)} \\ \Gamma(z,z) = \Gamma(x,x) - \Gamma(y,y) + 2i\Gamma(x,y), \\ \Gamma(\bar{z},\bar{z}) = \Gamma(x,x) - \Gamma(y,y) - 2i\Gamma(x,y) = \overline{\Gamma(z,z)} \\ \Gamma(z,\bar{z}) = \Gamma(x,x) + \Gamma(y,y). \end{cases}$$

Once this is expressed in terms of the variables x and y (and may be other variables), we may change again $x = (z+\bar{z})/2$, $y = (z-\bar{z})/(2i)$.

Then, we may apply the change of variable formula to any analytic function of z and $\bar{z}$ (such as polynomials).

This trick turns out to be very useful in many situations, as we shall see below, for example for Hermite or Jacobi polynomial representations.

3.4 Reversible measures

In general, one looks for some good reference measure. A natural candidate is an invariant measure μ, which satisfies $\int \mathrm{L}(f)d\mu = 0$, for any good function f. When this measure is finite, we chose it to be a probability and one expects that $P_t f \to \int f d\mu$ when $t \to \infty$. This is called sometimes the ergodicity of the process. Of course, when the measure is infinite, there is no such interpretation in general.

For example for the standard Brownian motion in $\mathbb{R}^d$, where $\mathrm{L} = \frac{1}{2}\Delta$ and μ is the Lebesgue measure dx, one has $(2\pi t)^{d/2}P_t f \to \int f dx$.

In reasonable situation, there is a unique such measure, up to a normalizing constant. If one wants to look for a density $\rho(x)$ of such a measure with respect to the Lebesgue measure, one solves the adjoint equation $\mathrm{L}^*(\rho) = 0$, where L^* is the adjoint of L with respect to the Lebesgue measure.

For example in the simplest case in $\mathbb{R}^d$ where $\mathrm{L} = \Delta$, then the unique solution is the Lebesgue measure, up to a constant.

In general, it is not easy to solve explicitly this equation (except in the particular case of symmetric operators), and the uniqueness amounts to determine the existence or non existence of positive harmonic functions, that is positive functions h such that $\mathrm{L}(h) = 0$.

The name invariant refers to the fact that if the law of ξ_0 is μ, then the law of ξ_t remains μ for any positive time t.

In what follows, we shall be interested in the symmetric case. We assume that the algebra $\mathcal{A}$ is included in $\mathcal{L}^2(\mu)$. Then, symmetry refers to the fact that, for any pair (f, g) of functions in $\mathcal{A}$,

$$\int f\mathrm{L}g d\mu = \int g\mathrm{L}f d\mu.$$

If one assumes that the constant functions belong to $\mathcal{A}$ (or that the constant may be reasonably approximated by functions in $\mathcal{A}$), applying the previous to $g = 1$ shows that if symmetry occurs, then the measure is invariant. We sometimes call it a reversible measure, since in this case, invariance is reinforced in the following property : for any time $t > 0$, the law of the process $(\xi_s, 0 \leq s \leq t)$ is the same as the law of $(\xi_{t-s}, 0 \leq s \leq t)$, whenever the law of ξ_0 is μ.

When μ has a smooth positive density ρ with respect to the Lebesgue measure, one has with smooth positive density measure $\rho(x)$ with respect to the Lebesgue measure, then L may be written as

$$\mathrm{L}(f) = \frac{1}{\rho}\sum_i \partial_i\Big(\rho \sum_j g^{ij}\partial_j f\Big), \tag{3.2}$$

as is readily seen using integration by parts in Ω, see [1].

From this, one may describe the operator uniquely from the matrix $(g^{ij}$ and ρ, since

$$b^i = \sum_j g^{ij} \log \rho + \sum_j \partial_j g^{ij}. \tag{3.3}$$

This allows to determine ρ, up to a constant, from b^i and g^{ij}, although in practice this is not always that easy.

We see in this way that the operator L in this case is entirely determined by Γ (that is the matrix (g^{ij}))s and ρ. On a more abstract level, this relies on the integration by parts formula.

In the symmetric case, one has

$$\int fL(g)d\mu = -\int \Gamma(f,g)d\mu. \tag{3.4}$$

In particular, from the positivity of Γ (the semi-ellipticity property), one has

$$\int f\mathrm{L}(f)d\mu \leq 0. \tag{3.5}$$

Applying this to an eigenvector, this shows why the eigenvalues $-\lambda_n$ of the operator L (when they exist) are always negative.

Moreover, the integration by parts formula (3.4) shows that the knowledge of Γ and μ allows to determine the operator L.

In practice, on some open set $\Omega \subset \mathbb{R}^d$, the integration by parts formula (3.4) remains true as long as f and g are smooth and one is compactly supported in Ω. In order to extend it to wider classes, one requires boundary conditions on the functions, such as Dirichlet (functions vanishingg at the boundary) or Neuman (normal derivatives vanishing at the boundary).

When μ has a density ρ with respect to the Lebesgue measure, we already saw an expression for its density ρ. When the matrix g is non degenerate, there is also another representation. The matrix g is then considered as a co-metric : its inverse is a Riemannian metric. That is, writing the inverse matrix (g_{ij}), we may consider, for some vector field V with coordinates $(V^i(x))$ in Ω, its length which is defined as $|V|^2(x) = \sum_{ij} g_{ij}(x)V^i(x)V^j(x)$. The matrix $g = (g^{ij})$ itself is then considered as a co-metric, that is a metric on one-forms. For example, for the one form df, $|df|^2 = \Gamma(f,f)$.

We introduce the Laplace operator Δ_g associated with g (supposed to be elliptic). This is the operator L that we obtain when we chose $\rho = \det(g)^{-1/2}$, which is the Riemann measure associated to the metric g. The main property of these Laplace operators (and that explains why they play a particular rôle in the theory, is that a Laplace operator remains a Laplace operator under change of coordinates (or local diffeomorphism). We shall come back to this property in Section 3.6

Then, the symmetric operator L is written as

$$L(f) = \Delta_g(f) + \Gamma(\log\rho_1, f), \tag{3.6}$$

where this time ρ_1 is the density with respect to the Riemann measure. This is a good way to track the transformation of a given L through a change of variables. Indeed, the inverse matrix g_{ij} may in this case considered as a Riemannian metric, and the above operator is the Laplace operator associated with this metric.

When the operator is symmetric, we may expect (it is not always the case, and this argument mainly works when the measure is finite, and even though, this is not yet always true), that it may be diagonalized in through a complete orthonormal basis of $\mathcal{L}^2(\mu)$, that is that there exists a complete orthonormal sequence (f_n) and some parameters λ_n such that $L(f_n) = -\lambda_n f_n$. From formula (3.5), we conclude that $\lambda_n \leq 0$.

The heat kernel is then entirely characterized through $P_t(f_n) = e^{-t\lambda_n} f_n$.

In this case, one has (at least formally), we already provided the representation of the heat kernel P_t as

$$P_t(f)(x) = \int p_t(x,y)f(y)\mu(dy),$$

where

$$p_t(x,y) = \sum_n e^{-\lambda_n t} f_n(x) f_n(y).$$

However, we do not know yet that this series is convergent, and even worse, this situation requires that the spectrum of L is discrete, which is not always the case. Moreover, this representation is made from oscillating functions, and the result in the end is non negative. For many purposes, it is not always convenient. Moreover, it is quite rare that one knows the eigenvalues λ_n, and even less the eigenvectors f_n (although there are many numerical methods which provide good approximation form them in low dimension).

We shall be particularly interested in the case when $\mathcal{L}^2(\mu)$ admits a complete orthonormal basis $P_q(x), q \in \mathbb{N}$, of polynomials such that $\mathrm{L}(P_q) = -\lambda_q P_q$, for some real (indeed non negative) parameters λ_q : that is, the spectrum is discrete and the eigenvectors are polynomials. This is equivalent to the fact that there exists an increasing sequence $\mathcal{P}_n$ of finite dimensional subspaces of the set $\mathcal{P}$ of polynomials such that $\cup_n \mathcal{P}_n = \mathcal{P}$ and such that L maps $\mathcal{P}_n$ into itself. In this case, one has a way to explicitly compute (at least recursively) all the eigenvalues and eigenvectors. This does not mean that one has compact formulas for the summation of the series, except in special cases.

In order to pursue the comparison with the finite state space, when L is replaced by a matrix L_{ij}, an invariant measure is a dual vector μ_i, where $\mu_i = \mu(i)$, satisfying

$$\forall j = 1, \cdots, n, \ \sum_i \mu_i L_{i,j} = \mu_j,$$

that is an eigenvector for the transposed matrix L^* with eigenvalue 1, whereas the matrix L has always an eigenvector with eigenvalue 1 which is the constant vector $f_i = 1$.

The reversibility however is insured as soon as, for any pair (i,j), $\mu_i L_{ij} = \mu_j L_{ji}$, so that the measure μ is determined up to a constant by the ratios $\frac{L_{ij}}{L_{ji}}$.

One sees immediately that reversible measure are invariant, but reversibility requires some special structure on the matrix L, namely that the ratios $\rho_{ij} = \frac{L_{ij}}{M_{ji}}$ satisfy the co-cycle property $\rho_{ij}\rho_{jl} = \rho_{il}$.

3.5 More about the algebra $\mathcal{A}$

Up to now, we did not describe very much the algebra $\mathcal{A}$. In the symmetric case, we want the integration by parts formula to be valid on it, but in practice this is not enough. If one wants to be able to describe P_t from the knowledge of L on $\mathcal{A}$, we want $\mathcal{A}$ to be dense in the domain of L, in the $\mathcal{L}^2(\mu)$ sense for example. For this we need to describe what is the domain of L : this is the set of functions in $\mathcal{L}^2(\mu)$ such that $\frac{1}{t}(P_tf - f)$ has an $\mathcal{L}^2(\mu)$ limit when $t \to 0$ (that is the domain of derivative of P_t at $t = 0$). The derivative at $t = 0$ is then $\mathrm{L}f$ (which is almast obvious from the representation $P_t = \exp(t\mathrm{L})$. The general theory of semigroups in a Banach space show that this domain is always dense in $\mathcal{L}^2(\mu)$.

Then, we want the algebra $\mathcal{A}$ to be dense in the domain, for the domain topology. That is, for any f in the domain, there exists a sequence of functions (f_n) in $\mathcal{A}$ such that $\lim_n f_n = f$ and $\lim_n \mathrm{L}f_n = \mathrm{L}f$. In this case, we say that $\mathcal{A}$ is a core, or that L is essentially self adjoint on $\mathcal{A}$.

Unfortunately, in general we do not know this domain. So that we are looking for conditions to insure that the algebra $\mathcal{A}$ is dense in the domain, without any knowledge of it, and even no knowledge of P_t itself, since we want to describe P_t from L on $\mathcal{A}$.

There are a few criteria for that. For example, when L is elliptic, we already mentioned the Riemannian structure associated to it. Then, if Ω is complete for this Riemannian structure, then the set of smooth functions compactly supported in Ω is a core for L. This completeness property may be translated in our langage in the following : there exist a sequence (f_n) of elements of $\mathcal{A}$, uniformly bounded, increasing to 1, such that $\Gamma(f_n, f_n)$ converges uniformly to 0.

But essential self adjointness may come from other properties : for example on a bounded set if the density of the measure ρ converges fast enough to 0 or to ∞. (This may be made precise for example on an interval, but we shall not extend on that and refer to [1].)

Another way to insure self-adjointness is when we may be sure that the operaor L diagonalizes in $\mathcal{A}$, that is when one may find in $\mathcal{A}$ a complete system of eigenvectors. Then, moreover, $\mathcal{A}$ is stable under P_t.

In many models described below, this diagonalisation will be realized through polynomials. But in any bounded open domain for example, the integration by parts formula may not be realized for polynomials, since in the process of integration by parts, boundary terms may occur. For this to happen, one requires an extra condition to hold. Assume for example that the domain $\Omega \subset \mathbb{R}^d$ has a piecewize $\mathcal{C}^1$ boundary, such that Stokes formula apply. Assume moreover that the measure is a probability measure with smooth density $\rho(x)$ in Ω. Then, a necessary and sufficient condition for the integration by parts to hold for on Ω for polynomials (or equivalently for the restrictions to Ω of smooth functions, is that the matrix g^{ij} degenerates at the boundary and that at any regular point of the boundary, the normal vector $n = (n_i)$ is in the kernel of it. More precisely, this requires that, for any regular point $x \in \partial\Omega$, one has

$$\forall i = 1, \cdots d, \ \sum_j \rho(x) g^{ij} n_j = 0.$$

3.6 Images of a diffusion operator

3.6.1 Closed systems

It may happen that one find a finite set of functions $(y_1, \cdots, y_p)$ such that, writing $y = (y_1, \cdots, y_p)$, $\mathrm{L}(y_i) = B_i(y)$, and $\Gamma(y_i, y_j) = G^{ij}(y)$. In this situation, we say that $(y_1, \cdots, y_p)$ form a closed system.

It may happen that the formula is closed for Γ and not for L : in this situation, we shall say that we have a closed system for Γ.

Then, for any smooth function $\Phi : \mathbb{R}^p \mapsto \mathbb{R}$, one has

$$\mathrm{L}(\Phi(y)) = \mathrm{L}_1(\Phi)(y),$$

where

$$\mathrm{L}_1 = \sum_{ij} G^{ij}(y) \partial^2_{ij} \Phi + \sum_i B_i(y) \partial_i(\Phi).$$

Then, L_1 is a new diffusion operator. In this situation, the image of the diffusion process ξ_t under $y : \Omega \mapsto \Omega_1 = y(\Omega) \subset \mathbb{R}^p$ is a new diffusion process ξ_{1t}, with generator L_1.

Whenever L is symmetric with respect to a reversible probability measure μ, then L_1 is symmetric with respect to a new reversible probability measure μ_1, which is the image of μ under the map y. Thanks to formula (3.3), this is often an efficient way to compute image measures.

3.6.2 Image under diffeomorphisms : change of variables

A first example of the previous situation appears when $y : \Omega \mapsto \Omega_1$ is a diffeomorphism. Then, the system is automatically closed setting $B_i = \mathrm{L}(y_i) \circ y^{-1}$ and $G^{ij} = \Gamma(y_i, y_j) \circ y^{-1}$.

It turns out that the image of a Laplace operator under such a diffeomorphism is again a Laplace operator (this is not the case for a generic image).

Moreover, the decomposition $\Delta_g + \Gamma(\log \rho, \cdot)$ is unchanged under diffeomorphisms (or local diffeomorphisms). This is the main advantage of the geometric language. In particular, if one knows how to compute the operator Γ in the new coordinates, we also know how to compute the image of the Riemann measure, and the image of a density with respect to the Riemann measure. This may be useful even in Euclidean cases (we shall see that later dealing with characteristic polynomials), since it avoids the computation of the Jacobian.

For example, in $\mathbb{R}^2$, in polar coordinates, with $(x,y) = re^{i\theta}$, $(r = \sqrt{x^2+y^2}, \theta = \arctan(y/x))$, then $\Gamma(r,r) = 1$, $\Gamma(\theta,\theta) = 1/r^2$, $\Gamma(r,\theta) = 0$, so that the image of the Lebesgue measure is $(\det g)^{-1/2} dr d\theta = r dr d\theta$, and the Laplace operator in polar coordinates writes

$$\partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r}\partial_r.$$

3.6.3 Products and wrapped products

Products is the first trivial thing to do with semigroups. When considering two independent processes ξ_t^1 and ξ_t^2, with generators L^1 and L^2, the product (ξ_t^1, ξ_t^2) is a Markov process and the generator, acting on a function $f(x,y)$ is $L_x^1 + L_y^2$.

A such, the d dimensional Brownian motion is just d independent copies of a 1-dimensional Brownian motion.

We may also often consider skew products (or wrapped products in Riemannian geometry). This consists in considering $L_x + g(x)L_y$. The associated process (ξ_t, ζ_t) is such that ξ_t is a Markov process and ζ_t is a a Markov process with generator L_y, but turning at a speed which a a function of ξ_t. The image of a wrapped product under $(x,y) \mapsto x$ is the process with generator L, but this is no longer the case for the image under $(x,y) \mapsto y$. This is for example the case of the 2 dimensional Brownian motion in polar coordinates.

Example of a wrapped product : the Euclidean Laplace operator of $\mathbb{R}^d$ seen in polar coordinates

$$\partial_r^2 + \frac{d-1}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}},$$

where $\Delta_{\mathbb{S}}$ is the spherical Brownian motion which is described below.

3.6.4 Invariance under transformations

Let Φ a transformation $X \mapsto X$. It may happen that the $\mathrm{L}\Big(f(\Phi)\Big) = \mathrm{L}(f)(\Phi)$. This means that the law of $\Phi(\xi_t)$ when $\xi_0 = x$ is the same than the law of ξ_t when $\xi_0 = \Phi(x)$.

For example, the Brownian motion is invariant under translation, but also under rotations.

If such is the case, functions which are invariant under the transformation Φ remain invariant after the action of L. When we find a set of such invariant functions which generate (as σ-algebras) all the invariant functions, then we get an image process.

Invariance may occur under the action of a vector field V. Then, the operator is also invariant under the diffeomorphisms $\exp(tV)$. A simple example of this situation is the Brownian motion in $\mathbb{R}^d$ under the translation along a coordinate $x \mapsto x + te_d$, where e_d is the last basis vector. Then, the generaor (Δ) commutes indeed with the vector field $\frac{\partial}{\partial x_d}$. A basis for invariant functions are then the functions $(x_1, \cdots, x_{d-1})$, which form a closed system, and the result of the projection is nothing else than the $d-1$-dimensional Brownian motion.

If we use the invariance of the Laplace operator with the rotations (that is with the full family of vector fields $V_{ij} = x_i\partial_j - x_j\partial_i$, we get as invariant function the radius $r = \sqrt{\sum_1^d x_i^2}$, and the result is nothing else than the Bessel process, with generator on $\mathbb{R}_+$

$$\partial_r^2 + \frac{d-1}{r}\partial_r.$$

We may also use some discrete group of transformations. For example, in $\mathbb{R}$, let's look at the reflection around 0. The image of the Brownian motion is the reflected Brownian motion. Then, a function is invariant under this reflection if and only if it is a function of $y = x^2$. We have $\Delta(x^2) = 2$ and $\Gamma(x^2, x^2) = 4x^2$, such that the operator becomes, with $y = x^2$

$$4y^2 f'' + 2f'.$$

Of course, going back to the usual coordinate $y = \sqrt{y}$ the usual BM (which is no surprise, it is a local diffeormorphism)

We shall describe many such examples of images, with some operators which are invariant under a big group action. Then, any subgroup generates a closed system of invariants, and provides an image. Using the rich theory of invariants provides many interesting models, as we shall see below.

4 Basic models in Euclidean spaces

The most common models for symmetric diffusions in $\mathbb{R}^d$ arise when considering it as Euclidean space. There is then a close connection between the Euclidean structure and the process, described through it's generator.

1. **Classical Brownian motion in $\mathbb{R}^d$**
   This process describes d independent one dimensional Brownian motions. Actually, we do not really describe the usual Brownian motion B_t here, but rather B_{2t}, which is much more convenient for further purpose. It's invariant measure is the Lebesgue measure, that is, up to a normalizing constant, the unique Radon measure which is invariant under translation.

   $$\Gamma(x_i, x_j) = \delta_{ij}, \mathrm{L}(x_i) = 0;\ \mathrm{L} = \Delta.$$

   Then,

   $$P_t(f)(x) = F(t, x) = \mathbb{E}\Big(f(x + \sqrt{2t}X)\Big),$$

   where X is a standard Gaussian variable $N(0, 1)$, that is the law of X is

   $$\frac{1}{(2\pi)^{n/2}}\exp(-\frac{1}{2}\|x\|^2)dx.$$

It is easily seen through integration by parts that for any smooth compactly supported fonction, the above expression solves the heat equation $\partial_t F = \Delta F$. The reversible measure is the Lebesgue measure, which has infinite mass, but the spectrum is not discrete.
A Brownian motion lives naturally in an Euclidean space. The generator is easily described on linear forms. For linear forms α_i,

$$\Gamma(\alpha_1, \alpha_2) = \alpha_2 \cdot \alpha_2, \mathcal{L}(\alpha_i) = 0.$$

This generator is invariant under rotations, symmetries and translations.
One may then look at the Brownian motion on subspaces. For example, projected onto the subspace orthogonal to e with $\|e\| = 1$, we get

$$\Gamma(\alpha_1, \alpha_2) = \alpha_2 \cdot \alpha_2 - (\alpha_1 \cdot e)(\alpha_2 \cdot e).$$

Then for example in $\mathbb{R}^n$, for this Brownian motion projected on the subspace $\sum_i x_i = 0$,

$$\mathrm{L}(x_i) = 0, \Gamma(x_i, x_j) = \delta_{ij} - \frac{1}{n}.$$

Imagine now that one has a process in $\mathbb{R}^n$ with these data, then, with $l = \sum_i x_i$,

$$\mathrm{L}(l) = 0, \Gamma(l, l) = 0.$$

If the process starts from some point $X_0 \in \mathbb{R}^n$, then, for any smooth compactly supported function $\mathbb{E}(f(l(X_t)) = f(l(X_0))$ so that $l(X_t) = l(X_0)$, and therefore it stays on the hyperplane $l(x) = a$ where it started from.
We shall recover a lot of such situations (in matrices, Lie groups, etc). Observe that then we may use the overdetermined system $(x_1, \cdots, x_n)$ on this hyperplane $\sum_i x_i = 0$, considering the functions x_i as the restrictions to $\sum_i x_i = 0$ of the linear forms $x \mapsto x_i$.
We saw that we may project the Euclidean Brownian motion onto any $d-1$ dimensional hyperplane : it reflects the invariance of the generator with respect of the translations, or its commutation with the vector fields $\sum_i \alpha_i \partial_{x_i}$, where α_i are constant.
We may also project it onto radial functions, with reflects the commutation with the infinitesimal rotations $x_i\partial_j - x_j\partial_i$. We have, with $r = \sqrt{\sum_i x_i^2}$ or $R = r^2$ are exactly those functions invariant under rotations. We have $\Gamma(R, R) = 4R$ and $L(R) = 2d$, so that it gives

$$LF = 4Rf'' + 2dF'$$

or, going to $r = \sqrt{R}$, gives the the Bessel processes

$$\partial_r^2 + \frac{d-1}{r}\partial_r.$$

2. **Euclidean complex Brownian motion : use of complex coordinates.**
   We already saw the use of complex coordinates in OU processes. For the Brownian motion in $\mathbb{R}^{2d} = \mathbb{C}^d$, one has
   $\Delta(z_i) = \Delta(\bar{z}_i) = 0$, $\Gamma(z_i, z_j) = 0, \Gamma(z_i, \bar{z}_j) = 2\delta_{ij}$.
   For example, for the complex Brownian motion, of any analytic function of (z_j) one has $\mathrm{L}(f(z)) = 0$. We then have the conformal martingales, that is if ξ_t is a Brownian motion and f is an holomorphic function, then $f(\xi_t)$ is a martingale.
3. **Ornstein-Uhlenbeck in $\mathbb{R}^d$**
   This process has as invariant measure the standard $N(0, \mathrm{Id})$ Gaussian measure on $\mathbb{R}^d$.
   $$\Gamma(x_i, x_i) = \delta_{ij}, \mathrm{L}(x_i) = -x_i,$$
   or
   $$s\mathrm{L} = \Delta - x.\nabla.$$
   The reversible measure is the standard Gaussian measure. Once again, one has an explicit expression for the heat kernel as
   $$P_t(f)(x) = \mathbb{E}\Big(f(e^{-t}x + \sqrt{1-e^{-2t}}X)\Big).$$
   The spectrum is discrete, the eigenvectors are the Hermite polynomials, and the eigenvalues are n.
   The generator commutes with rotation, but not with translations. One may also look at Ornstein-Uhlenbeck operators with other Gaussian invariant measures, for example with variance σ^2, and in this situation we get the same Γ but now with $\mathrm{L}(x_i) = -\frac{1}{\sigma^2}x_i$. Letting σ go to ∞ we obtain an approximation of Brownian motion, but with a finite invariant measure.
   As for Brownian motion, and Ornstein-Uhlenbeck process lives naturally on an Euclidean space, with, for the same carré du champ as a Brownian motion, and for any linear form α, $\mathrm{L}(\alpha) = -\alpha$.
   We shall come back to this example in Section 7.
4. **Spherical brownian motion on $\mathbb{S}^{d-1} \subset \mathbb{R}^d$**
   This reversible process has invariant measure the uniform measure o the unit sphere, that is the unique probability measure which is invariant under rotations.
   $$\Gamma(x_i, x_j) = \delta_{ij} - x_i x_j, \mathrm{L}(x_i) = -(d-1)x_i.$$
   There is not simple formula for the heat kernel. The reversible measure is the uniform measure on the sphere, and the spectrum is discrete. The eigenvalues are $n(n+d-1)$. The eigenvectors are the restriction to the sphere of the Harmonic homogenous polynomials with degree n.
   Once again, it exists on any Euclidean space, with, for two linear forms α_1 and α_2 and in dimension d for the Euclidean space
   $$\Gamma(\alpha_1, \alpha_2) = \alpha_1 \cdot \alpha_2 - \alpha_1\alpha_2,\ \mathrm{L}(\alpha_1) = -(d-1)\alpha_1$$

How do we obtain those formulas ? There are three ways of considering the spherical Brownian motion. One may think of the sphere $\mathbb{S}^{d-1}$ as embedded in $\mathbb{R}^d$ and decide that to compute the Laplace operator of a function f defined on $\mathbb{S}^{d-1}$, one may first extend it into a function in the neighborhood of $\mathbb{S}^{d-1}$ into a functions which does not depend on the radius, then compute the Euclidean Laplace extension of this extension and then restrict it to the sphere. Doing this, one seems that the extension of the coordinate x_i is $\frac{x_i}{|x|}$, and this leads to the announced result.
Next, one may consider the sphere itself as a Riemannian manifold, the unit ball $\mathbb{B}_{d-1}$ as a local chart of the upper (or lower) half sphere. If $x = (x_1, \cdots, x_{d-1}) \in \mathbb{B}_{d-1}$, it corresponds to a point on the sphere $(x, \pm\sqrt{1-|x|^2})$. Then, given a vector $V = (V_1, \cdots, V_{d-1})$ at some point $x \in \mathbb{B}_{d-1}$, if consider it as the projection of a tangent vector on the sphere, we see that it's real length is $\sum_{ij} g_{ij} V_i V_j$, where

$$g_{ij} = \frac{x_i x_j}{1-|x|^2} + \delta_{ij}.$$

This gives on the unit ball $\mathbb{R}^{d-1}$ a Riemannian metric g_{ij} whose inverse is $g^{ij} = \delta_{ij} - x_i x_j$. Then, we may consider the Laplace operator associated with this as given in formula (3.6).
Moreover, we may also consider it as a homogeneous space, that is a space on which the group $SO(d)$ acts transitively (more precisely as $SO(d)/SO(d-1)$, $SO(d-1)$ being considered as the set of orthogonal transformations which lives the point $(1, 0, \cdots, 0)$ invariant.
The infinitesimal rotation in the plane (x_i, x_j) is the operator $D_{ij} = x_i\partial_j - x_j\partial_i$. Indeed, with two coordinates (x_1, x_2), let R_t be the rotation $R_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. Then, $\partial_t f(R_t x)\mid_{t=0}$ is this operator. Now, it turns out that, although the various D_{ij} do not commute to each other, $\sum_{i<j} D_{ij}^2$ commute which each of the D_{ij} and, up to some scalar constant, it is gain the spherical Laplace operator.
How may one device from the formulas that the operator lives on the unit sphere ? If one computes for the spherical operator $L(R)$, where $R = \sum_i x_i^2$, then one finds

$$L(R) = \sum_1^d -2(d-1)x_i^2 + 2\sum_1^d (1-x_i^2) = 2d(1-R),$$

which is 0 on $R^2 = 1$. Similarly $\Gamma(R, R) = 4\sum_{ij} x_i x_j(\delta_{ij} - x_i x_j) = R(1-R)$, one again vanishing on $R = 1$.
It is not enough to assert that the operator defined by these relations live on on the sphere. But is we consider in $\mathbb{R}^d$ the following operator

$$\mathrm{L}(x_i) = -(d-1)x_i, \Gamma(x_i, x_j) = \delta_{ij}|x|^2 - x_i x_j,$$

then one gets $L(R) = \Gamma(R, R) = 0$.

From this, $L(F(R)) = 0$ for any function F and $F(R(\xi_t)$ is a martingale. Therefore $|\xi_t|$ is constant and the process stays on the sphere where it started from.
The reversible measure for $\Delta_{\mathbb{S}}$ is the uniform measure on the sphere, that is the unique probability measure which is invariant under rotations.
As imbedded in $\mathbb{R}^d$, introducing the Euler operator $V = \sum_i x_i \partial_i$, then

$$\Gamma_{\mathbb{S}}(f,f) = \Gamma_{\mathbb{E}}(f,f) - (Vf)^2, \; \Delta_{\mathbb{S}}(f) = \Delta_{\mathbb{E}}(f) - V^2(f) - dV(f).$$

About projections, the first observation is that it projects on the unit ball in $\mathbb{R}^p$ through $x \mapsto (x_1, \cdots, x_p)$, into a process which has the same Γ but with a different drift. The invariant measure for this projection is $(1-R)^{(d-1-p)/2-1}dx$. This provides the image measure of the uniform measure on the sphere onto this p-space. For $p = d-1$, then the measure is $(1-R^2)^{-1/2}$, which is therefore the Riemannian measure in these coordinates (the unit ball $\mathbb{B}_{d-1}$ being a local chart for the upper half-sphere $\mathbb{S}^{d-1}$. For $p = 1$, we get the measure $(1-x^2)^{(d-1)/2-1}$, and the associated process is the symmetric Jacobi operator $(1-x^2)\partial^2 - (d-1)x\partial_x$, for which the eigenvectors are the symmetric Jacobi (or Gegenbauer, or ultrasherical) operator (see below).
One may also observe

$$(X_1 = x_1^2 + \cdots + x_{p_1}^2, Y_2 = x_{p_1+1}^2 + \cdots + x_{p_1+p_2}^2, \cdots, Y_k = x_{p_1+\cdots+p_{k-1}+1}^2 + \cdots + x_{p_1+\cdots+p_k}^2),$$

with $p_1 + \cdots + p_k \leq n$.
One has

$$L(Y_i) = -2dY_i + 2p_i$$

and

$$\Gamma(Y_i, Y_j) = 4Y_i(\delta_{ij} - Y_j).$$

This gives a process on the simplex $\{Y_i \geq 0, \sum_i Y_i \leq 1\}$, with invariant measure the Dirichlet law

$$Y_1^{p_1/2-1} Y_1^{p_2/2-1} \cdots Y_k^{p_k/2-1} (1 - Y_1 - \cdots - Y_k)^{p_{k+1}/2-1} dY_1 \cdots dY_k$$

with $p_1 + \cdots + p_k + p_{k+1} = d$.
Once again, for $k = 1$ we recover a dissymmetric Jacobi process (see below), with invariant measure the beta law $x^a(1-x)^b dx$, and eigenvectors the Jacobi polynomials.
Once again, these Dirichlet laws may be seen as the image of the uniform measure on the sphere. Moreover, one may represent a random point in $\mathbb{S}^{d-1}$ as $(\rho_1 U_1, \cdots, \rho_k U_k)$, where $(\rho_1^2, \cdots, \rho_k^2) = (Y_1, \cdots, Y_k)$ which law has just been described, and $U_1, \cdots, U_k$ belong to unit spheres in $\mathbb{S}^{p_1-1}, \cdots, \mathbb{S}^{p_k-1}$, and are moreover uniform and independent.

5. **Hyperbolic Brownian motion**

Although we shall not really use it later, it is worth to mention the third basic geometric model.

The hyperbolic Laplace is the Riemannian structure on the hyperboloid

$$\mathbb{H}_d = \{x_1^2 + \cdots + x_d^2 + 1 = x_{d+1}^2\} \subset \mathbb{R}^{d+1},$$

when we put on it the Riemannian structure inherited from the quadratic form $\sum_1^d x_i^2 - x_{d+1}^2$.

In fact, this hyperboloid has two connected components, and we chose the one with $x_{x+1} > 0$.

On the surface, we have $\sum_1^{d+1} x_i dx_i - x_{d+1}dx_{d+1} = 0$, so that the metric on the surface, which is $\sum_i (dx_i)^2 - (dx_{d+1})^2$ may be rewritten as

$$\sum_1^{d+1}(dx_i)^2 - (\sum_i^{d+1} \frac{x_i}{x_{d+1}} dx_i)^2,$$

with $x_{d+1}^2 = 1 + r^2$, where $r^2 = \sum_1^d x_i^2$. That is the metric is

$$\Gamma(x_i, x_j) = \delta_{ij} - \frac{x_i x_j}{1+r^2},$$

and it's inverse is given, in this system of coordinates, by

$$g^{ij} = \delta_{ij} + x_i x_j.$$

We have $\det(g^{ij}) = 1 + r^2$, and the associated Laplace operator is determined by Γ and the invariant measure which is $\frac{1}{(1+r^2)^{1/2}} dx$, which provides

$$\Gamma(x_i, x_i) = \delta_{ij} + x_i x_j, \ \Delta_{\mathbb{H}}(x_i) = dx_i. \tag{4.7}$$

It is invariant under the elements of $O(d, 1)$ (leaving invariant the quadratic form $\sum_i^d x_i^2 - {}_{d+1}^2$.

This is completely similar to the spherical Laplace operator in the ball.

There are many other representations of this hyperbolic Laplace operator. Through a change of variable in the unit ball, or in the upper-half space, which are the most commonly used, essentially because in these representations, the hyperbolic Laplace operator is conformal to the usual Euclidean one (we say that two operators are conformal to each other if their carré du champ operators Γ and Γ_1 satisfy $\Gamma = c(x)\Gamma_1$, for some function $c(x)$. This property is not preserved through a change of coordinates.

6. **Sturm Liouville operators in dimension 1**

On $\mathbb{R}$, all operators may be written as

$$\mathrm{L} = a(x)f'' + b(x)f'.$$

Now, when $a > 0$, one may change variables setting $\sqrt{a}\partial_x = \partial_y$, that is solving the differential equation $\frac{dy}{dx} = \frac{1}{\sqrt{a(x)}}$, to translate L into

$$\mathrm{L}_1 = \partial_y^2 + b_1(y)\partial_y.$$

This operator is always reversible for the measure $\exp(V(y))dy$, with $V' = b_1$.The stochastic process with generator L_1 is nothing else than $y(\xi_t))$, where ξ_t has generator L. So that symmetry is always true, and moreover, up to a change of variables, there exist only one metric.

On a compact interval, the description of the generator on, say, compactly supported functions, is not enough to describe the semigroup. One has to add boundary conditions. For example, on $(0, 1)$ with $Lf = \frac{1}{2}f''$, one has least two choices : the Dirichlet or the Neuman boundary conditions. It corresponds to the Brownian motion being killed or reflected at the boundary. In the first case, the eigenvectors are $\sin n\pi x$, in the second one $\cos(n\pi x)$. The fact that the description of the operator on compactly supported functions is enough to describe a unique self-adjoint operator is called essential self-adjointness. It happens as soon that the space is complete, or as soon that the drift is too repelling at the boundary. We shall not enter into such details here since we shall impose the eigenvectors in most cases, such describing in a intrinsic way the self-adjoint extension we are dealing with, most of the time Neuman.

7. **The three families of orthogonal polynomials in dimension 1**

In dimension 1, their exist only three families of diffusion with polynomial eigenvectors, up to an affine transformation, corresponding to Hermite, Laguerre and Jacobi polynomials.

(a) The Ornstein-Uhlenbeck operator (OU in short in what follows) : $\mathcal{L}(f) = f'' - xf'$, symmetric with respect to the gaussian measure $\gamma(dx) = (2\pi)^{-1/2}e^{-x^2/2}dx$. It has as eigenvectors the Hermite polynomials H_n, which satisfy $\mathcal{L}(H_n) = -nH_n$. Thee Hermite polynomials are the natural b-orthonormal basis of $\mathcal{L}^2(\gamma)$ which is obtained by the standard orthonormalization of the sequence $\{1, x, x^2, \cdots, x^n \cdots\}$.

The use of complex coordinates may be quite useful. Then, for example, one has the following representation of the unnormalized Hermite polynomial $H_n(x)$

$$H_n(x) = c_n \int_{\mathbb{R}} (x + iy)^n \gamma(dy),$$

where $\gamma(dy)$ denotes the standard Gaussian measure on $\mathbb{R}$.

To see this, look at the two dimensional Ornestein-Uhlenbeck operator in the complex plane $\mathrm{L}(z) = -z$, and $\Gamma(z, z) = 0$. Then, $\mathrm{L}(z^n) = -nz^n$. Now, look at it in real coordinates as $\mathrm{L} = \mathrm{L}_x + \mathrm{L}_y$. One has

$$\mathrm{L}_x \int_{\mathbb{R}} z^n \gamma(dy) = \int_{\mathbb{R}} \mathrm{L}_x(z^n)\gamma(dy).$$

But, since $\gamma(dy)$ is the invariant measure for L_y, one also has

$$\int_{\mathbb{R}} \mathrm{L}_y z^n \gamma(dy) = 0,$$

from which one also gets

$$\mathrm{L}_x \int_{\mathbb{R}} z^n \gamma(dy) = \int_{\mathbb{R}} \mathrm{L} z^n \gamma(dy) = -n \int_{\mathbb{R}} z^n \gamma(dy),$$

from which we deduce that $\int_{\mathbb{R}} z^n \gamma(dy)$ is an eigenvector of L_x, and therefore proportional to the Hermite polynomial H_n.
Moreover, the structure of the operator L may also lead to recurrence formulas for the moments.
This is quite easy in the Ornstein-Uhlenbeck case. One has $\mathrm{L}(x^n) = -nx^n + n(n-1)x^{n-2}$, so that writing $\int \mathrm{L}x^n \gamma(dx) = 0$, one gets

$$\int x^n \gamma(dx) = (n-1) \int x^{n-2} \gamma(dx).$$

This simple trick may be extended far beyond this simple case, for example to get recurrence formulas for the moments of the spectrum of a symmetric Gaussian matrix, where the recurrence formulas appear to be much more difficult to obtain.

(b) The Laguerre operator (or squared radial generalized Ornstein–Uhlenbeck operator) on $I = \mathbb{R}_+^*$

$$L_a = x\frac{d^2}{dx^2} + (a-x)\frac{d}{dx}\,, \quad a > 0.$$

It is symmetric with respect to the measure $\mu_a(dx) = C_a x^{a-1} e^{-x}\,dx$. Its eigenvector are the Laguerre polynomials $(Q_n)_n$, with $\mathcal{L}_a(Q_n) = -nQ_n$. When a is a half integer, then if one considers the OU operator in $\mathbb{R}^n$, with generator $\mathcal{L} = \Delta - \sum_i x_i \partial_i$ (that is the sum of one dimensional OU operators acting separately on each variable), and consider the function $R = \sum_i x_i^2$, then

$$\mathcal{L}(F(R/2)) = 2\mathcal{L}_a(F)(R/2),$$

with $a = d/2$.
Then, the Laguerre operator with parameter $d/2$ is the image of the d dimensional OU operator.

(c) The Jacobi operator on $I = (-1, 1)$

$$J_{a,b} = (1-x^2)\frac{d^2}{dx^2} - \big(a - b + (a+b)x\big)\frac{d}{dx}\,, \quad a, b > 0.$$

It is symmetric with respect to the measure $\mu_{a,b}(dx) = C_{a,b}(1-x)^{a-1}(1+x)^{b-1}\,dx$, The Jacobi polynomials Q_n satisfy $J_{a,b}(Q_n) = -n(n+a+b-1)Q_n$.

In the same way that the Laguerre operator is an image of OU, then the Jacobi operator for half integers a and b are images of spherical Laplace operator. More precisely, on the sphere on dimension d, if one considers the function $2\sum_{i=1}^{p} x_i^2 - 1$, with values in $(-1,1)$, then the image of $\Delta_{\mathbb{S}^d}$ through it is $4\mathrm{L}_{p/2,(n-1-p)/2}$.
In the symmetric case $a = b$, there exists a simpler representation from the sphere : namely, looking at the spherical Brownian motion acting on the single variable x_1, one gets directly the Jacobi operator $J_{n/2,n/2}$.
Moreover, looking at the Jacobi for $a = b = n/2$, changing x into $y = \sqrt{n}x$, and letting n to ∞, then $\frac{1}{n}J_{n/2,n/2}$ converges to OU. In the same way, one may see Laguerre operators as limits of dissymmetric Jacobi operators, and indeed, all the important properties that we want to describe in dimension 1 are concerned with the Jacobi case (described later).
The complex variable trick used for the Hermite polynomial also works here, at least in the symmetric case (the non symmetric one is also at hand, but through much more complicated formulas). For this, set $a = b = n/2$, where n is a parameter. The two dimensional Jacobi operator acts on the unit ball as

$$\begin{cases} \Gamma(x,x) = 1 - x^2,\ \Gamma(y,y) = 1 - y^2,\ \Gamma(x,y) = -xy \\ \mathrm{L}(x) = -nx, \mathrm{L}(y) = -ny \end{cases}$$

When n is an integer, this is nothing else than the projection of the spherical Laplace operaor in two dimensions.
Setting $y = \sqrt{1-x^2}t$, where $t \in (-1,1)$, it turns out that this operator may be see as a skew product

$$J_{n/2,n/2,x} + \frac{1}{1-x^2} J_{(n-1)/2,(n-1)/2,t},$$

with reversible measure $\mu_{n/2,n/2}(dx)\mu_{(n-1)/2,(n-1)/t}(dt)$.
We just have to check that that

$$\Gamma(t,t) = \frac{1-t^2}{1-x^2}, \Gamma(x,t) = 0,\ \mathrm{L}(t) = \frac{-(n-1)t}{1-x^2}.$$

Then, with $z = x + iy$, one has $\mathrm{L}(z) = -nz$, $\Gamma(z,z) = -z^2$, so that

$$\mathrm{L}(z)^p = -p(n+p-1)z^p.$$

One may then obtain the representation, when $a = b = n/2$ and any integer p for the associated polynomial $J_p^{(n/2,n/2)}$

$$J_p^{(n/2,n/2)} = C_p \int_{(-1,1)} (x + i\sqrt{1-x^2}t)^p \mu_{(n-1)/2,(n-1)/2}(dt).$$

The construction we made above of the projection of the spherical Brownian motion on the 2 dimensional unit ball may be extend to larger dimensions.

Indeed, if one considers the projection of the d dimensional spherical Brownian motion ξ_t on $\mathbb{S}^d$, it may be decomposed into its projection on $\xi_t^{(1)} \in (-1,1)$, which is a symmetric Jacobi diffusion with parameter $a = b = d/2$, and we may then write $\xi_t = (\xi_t^{(1)}, \sqrt{1-(\xi_t^{(1)})^2}\zeta_t$, where $\zeta_t \in \mathbb{S}^{d-1}$. The generator $\Delta_{\mathbb{S}^d}$ in these coordinates $(x,\theta) \in (-1,1) \times \mathbb{S}^{d-1}$ may be written as

$$\Delta_{\mathbb{S}^d} = J_{d/2,d/2,x} + \frac{1}{1-x^2}\Delta_{\mathbb{S}^{d-1},\theta},$$

and one may repeat this construction with the Brownian motion on $\mathbb{S}^{d-1}$, ending up with a decomposition into embeded skew products of symmetric Jacobi operators. In particular, this allows to decompose the uniform measure on the sphere into direct products of Jacobi measures (with decreasing parameters).

8. **Diffusions on the simplex**

The d dimensional simplex $\mathbb{D}_d$ is the set of points $\{x_1, \cdots, x_d\}$ in $\mathbb{R}^d$ such that $x_i \geq 0$ and $\sum_1^d x_i \leq 1$. it could be seen as the set of probability measures on $d+1$ points, and as such plays an important rôle in many areas of probability, statistics and computer science. We already saw a diffusion process on the simplex as an image of the sphere. Indeed, there are much more such diffusion processes on the simplex which have the property that may be diagonalized through orthogonal polynomials. We already saw how one may construct such operators form the spherical Laplace operator, with

$$\Gamma(x_i, x_j) = x_i(\delta_{ij} - x_j),$$

the values of $\mathrm{L}(x_i)$ being determined by the value of Γ and the Dirichlet distribution.

One important property, as we shall see later in Section 5, is that for this Γ and, with $x_{d+1} = 1 - \sum_I^d x_i$, one has, for any $i = 1, \cdots, d$, any $j = 1, \cdots, d+1$,

$$\Gamma(x_i, x_j) = m_{ij} x_j, \tag{4.8}$$

where m_{ij} are degree 1 polynomials (this is obvious for $j = 1, \cdots, d$ but not when $j = d+1$). Here, the boundary of the domain $\mathbb{D}_d$ is determined by the equation $x_1 \cdots x_{d+1} = 0$.

There are many symmetric diffusion operators on the simplex withe the Dirichlet measure as invariant measure, where we recall that the Dirichlet measure is a probability measure on $\mathbb{D}_d$ with density $Cx_1^{a_1} \cdots x_{d+1}^{a_{d+1}}$, $a_i > 1$, $i = 1, \cdots, d+1$.

Indeed, the simplex is one of the few examples of domains in finite dimension for which there exist many Γ structures for which the boundary equation (4.8) has many different solutions. Namely, one may define a family of metrics depending on a symmetric $(d+1)\times(d+1)$ matrix $A=(A_{ij})$ with non negative entries as

$$g^{pq} := \Gamma_A(x_p, x_q) = -A_{pq}x_px_q + \delta_{pq}x_p\sum_{k=1}^{d+1}A_{pk}x_k, 1\leq p\leq q\leq d, \tag{4.9}$$

where in the previous equation x_{d+1} stands for $1-x_1-\cdots x_d$. The operator is elliptic on the simplex as soon as, for every $p\neq q$, $A_{pq}\neq 0$. One should check that the value of A_{ii} plays no rôle in the definition of Γ_A, and we shall set $A_{ii}=0$.
Indeed, one may check that ellipticity is insured as soon as the matrix A is recurrent in the sense of matrices with non negative entries, that is that for any $i\neq j$, there exists some parameter n such that $A^n_{ij}\neq 0$.
One may write the operator Γ_A as $\sum_{1\leq i<j\leq d+1}A_{ij}\Gamma_{ij}$, where
with

$$\Gamma_{ij}(x_p,x_q) = x_ix_j[\delta_{pq}(\delta_{pi}+\delta_{pj}) - (\delta_{pi}\delta_{qj}+\delta_{pj}\delta_{qi})], \tag{4.10}$$

All these metrics satisfy the boundary equation (4.8), as easily checked. Then, for the reversible Dirichlet measure with parameters (a_i), one gets

$$\mathrm{L}_{ij}(x_p) = (\delta_{pi}-\delta_{pj})(x_j(a_i+1)-x_i(a_j+1)). \tag{4.11}$$

Once again, when the parameters a_i are half integers, one has a geometric interpretation for these operators L_{ij} coming from spheres. Again, consider a sphere $\mathbb{S}^{N_1}\subset\mathbb{R}^d$, where we chose a partition of the set of indices $\{I_1,\cdots,I_{d+1}\}$ with size $|I_i|=p_i$. Now, for any pair of coordinates (y_p,y_q) in $\mathbb{R}^N$, one considers the infinitesimal rotation $D_{pq}=y_p\partial_{y_q}-y_q\partial_{y_p}$. Then we look at the operator

$$\Delta_{ij} = \sum_{p\in I_i, q\in I_j} D^2_{pq}.$$

For this operator, the variables $x_i=\sum_{p\in I_i}y_p^2$ form a closed system, and its image is $4\mathrm{L}_{ij}$, where the parameters a_i are $a_i=\frac{p_i}{2}-1$.

9. **Symmetric matrices Brownian motion**

The linear space of symmetric matrices is an Euclidean space, when endowed with the norm $\|M\|^2=\operatorname{trace}(M^2)$. On this Euclidean space,

$$\Gamma(m_{ij},m_{kl}) = \frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),\ \mathrm{L}(m_{ij})=0.$$

This is the Euclidean Brownian motion on the set of symmetric matrices with the Euclidean norm $\operatorname{trace}(M^2)$.
This contains the fact that we look at the Brownian motion on the set of symmetric matrices. If we start from a symmetric matrix, we stay on the set of symmetric matrices.

10. **Hermitian matrix Brownian motion**.

$$\Gamma(z_{ij}, z_{kl}) = 0,\ \Gamma(z_{ij}, \bar{z}_{kl}) = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

11. **Brownian motion on Riemannian manifold, Laplacians** : this is when the measure is $\det(\Gamma)^{-1/2}$. We already mentioned this very wide class, and its particular propert which is its invariance under diffeomorphisms.

12. **Brownian motion on semisimple compact Lie groups : the Casimir operator.**

Take a Lie group $\mathcal{G}$ (say a compact group of matrices). Then, the Lie algebra L is the set of matrices A such that $e^{tA} \in \mathcal{G}$. It is a linear space, stable under $(A, B) \mapsto [A, B] = AB - BA$.

One associates to A a vector field X_A on $\mathcal{G}$ by setting

$$X_A(f)(g) = \partial_{t|t=0} f(e^{tA} g).$$

(Left invariant vector fields.)

If A has entries A_{ij}, then, and g has entries m_{ij},

$$X_A(f) = \sum_{i,j,k} A_{ik} m_{kj} \partial_{m_{ij}} sf.$$

X_A maps polynomials of degree n in the entries m_{ij} into polynomials of the same degree.

We also have right invariant vector fields. A right invariant vector field commutes with a left invariant one : this reflects the associativity of the multiplication in the group.

From the definition of the Haar measure μ, (invariant under $g \mapsto g_1 g$, for any $g_1 \in \mathcal{G}$), then $X_A^* = -X_A$ in $\mathcal{L}^2(\mu)$, so that X_A^2 is symmetric. If $(A_1, \cdots, A_k)$ span the Lie algebra, then $\sum_k X_{A_k}^2$ is an elliptic operator, symmetric for the Haar measure. It is always a Laplacian, and maps polynomial in the entries into polynomial in the entries with the same degree.

Now, if the group is semisimple compact, then there is a natural quadratic form of the Lie algebra which is defined as such. For $A \in$ L, $B \mapsto [A, B] := ad(A)(B)$ is a linear operator on L, and $[ad(A), ad(B)] = ad([A, B]$. This is a representation of the Lie algebra. Then, consider the scalar product $\langle A, B \rangle = \text{trace}\,(ad(A)ad(B))$. One possible definition of semisimple is that this quadratic form is non degenerate. It turns out that it is then definite negative exactly when $\mathcal{G}$ is compact.

It turns out that, for semisimple compact Lie groups, if $(A_1, \cdots, A_n)$ is an orthonormal basis of L for this quadratic form, then $\mathcal{L} = \sum_i X_{A_i}^2$ commutes with the group action, from the right and from the left, and it is independant of the choice of the basis. This operator is called the Casimir operator. It is therefore a Laplacian, with Riemannian measure the Haar measure, and moreover it has constant

Ricci curvature 1/4. (Examples of Einstein manifolds). The Casimir operators map polynomials in the entries into polynomial in the entries, without increasing the degrees.

Examples of semisimple compact Lie groups are $SO(d)$, $SU(d)$, $Sp(d)$.

13. $SO(d)$ **Brownian motion** For an orthogonal matrix with entries m_{ij}, one gets

$$\begin{cases} L(m_{ij}) = -(d-1)m_{ij}, \\ \Gamma(m_{kl}, m_{qp}) = \delta_{(kl)(qp)} - m_{kp}m_{ql}. \end{cases}$$

We observe first that if we extract a sub matrix N selecting p lines and q columns, we obtain a process with the same relations, this time no longer living on the orthogonal group. This is the matrix Jacobi process of Y. Doumerc, developed by N. Demni.

Indeed, with this formula, for the projected matrix n (on the set of $p\times q$ matrices, we see that

$$\Gamma(\log(\det(I-nn^*), n_{ij}) = -2n_{ij},$$

and that, if ρ is the density, one also has

$$\Gamma(\log\rho, n_{ij}) = -(d-1-p-q)n_{ij}.$$

For which we deduce that if there is a density, it has to be $(\det(I-nn^*))^{(d-1-p-q)/2}$, and in fact the condition is $p+q\leq d$, in which case the image is the symmetric domain $nn^*\leq Id$. When it is not, (for example when the length of the columns are maximal), then it lives on a sub-manifold, always algebraic. In fact, this process on $p\times q$ matrices to have a density when the condition $p+q<d+1$ holds. It does nor matter too much here if we chose mm^* or m^*m those matrices having the same eigenvalues + a number of 0 for the biggest ones.

Observe that with $P = \det(\mathrm{Id}-mm^*)$ one has

$$\Gamma(\log P, \log P) = 4(-p + \mathrm{trace}\,(\mathrm{Id}-mm^*)^{-1})$$

.

$\Gamma(P,P) = PQ$, where Q is a polynomial in the entries of M. If we just select one column (or one line), this is the spherical process. One may then look at many different things.

If we consider the extracted $p\times q$ matrix n as before and $N = nn^*$, one gets a new operator on symmetric $p\times p$ matrices

$$\begin{cases} L(N_{ij}) = -2dN_{ij} + 2q\delta_{ij} \\ \Gamma(N_{ij}, N_{kl}) = \delta_{ik}N_{jl} + \delta_{il}N_{jk} + \delta_{jk}N_{il} + \delta_{jl}N_{ik} - 2N_{ik}N_{jl} - 2N_{il}N_{jk} \end{cases}.$$

Once again, one may consider its spectral measure, and obtain a new operator on the characteristic polynomial.

One has to be careful when using formula (3.2) to determine the density with respect to the Lebesgue measure, since one wants for example to work with a system of coordinates, that is here N_{ij} with $i \leq j$ and not all the N_{ij} (taking in account the symmetric structure of N). In this respect, $\partial_{N_{ij}} N_{kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}\delta_{ik}$.

$$\Gamma(\log \rho, N_{ij}) = L(N_{ij}) - \sum_{k \leq l} \partial_{N_{kl}} \Gamma(N_{kl}, N_{ij}),$$

one gets

$$\sum_{k \leq l} \partial_{N_{kl}} \Gamma(N_{kl}, N_{ij}) = 2(p+1)(\delta_{ij} - 2N_{ij}),$$

$$\Gamma(\log \rho, N_{ij}) = 2(q-p-1)\delta_{ij} + 2(2(1+p)-d)N_{ijs},$$

while

$$\Gamma(\log \det(\mathrm{Id} - N), N_{ij}) = -4N_{ij}.$$

We see that it is only when $q = p+1$ that we have a nice density $\det(I-N)^\alpha$. The problem here comes from the fact that the Lebesgue measure does not project nicely under $n \mapsto nn^*$.

The problem comes from the projection of the Lebesgue measure. How to find it ? We consider the flat brownian motion

$$\begin{cases} L(m_{ij}) = 0, \\ \Gamma(m_{ij}, m_{kl}) = \delta_{ik}\delta_{jl}. \end{cases}$$

in dimension $p \times q$, and consider as before $M_{ij} = \sum_q m_{ip} m_{jp}$ ($M = mm^*$). Then

$$\Gamma(N_{ij}, N_{pq}) = \delta_{ip} N_{jq} + \delta_{iq} N_{jp} + \delta_{jp} N_{iq} + \delta_{jq} N_{ip}.$$

with $L(N_{ij}) = 2q\delta_{ij}$.

Then, with $N_{ij}, i \leq j$ as a system of coordinates, one has

$$\sum_{a \leq b} \partial_{N_{ab}} \Gamma(N_{ab}, N_{ij}) = 2(p+1)\delta_{ij}$$

and therefore, of it has a density

$$\Gamma(\log \rho, N_{ij}) = 2(q-p-1)\delta_{ij}.$$

But, $\Gamma(\log \det(N), N_{ij}) = 4\delta_{ij}$, and this gives the image measure of the Haar measure $\det(N)^{(q-p-1)/2}$, which is locally integrable as soon as $q-p > -1$ (for integers, it gives $q \geq p$, which was to be expected, since otherwise, the image would be carried by sets of measure 0, they is the set of matrices with determinant 0). To make things rigorous, one should replace the Euclidean Brownian motion by the Ornstein-Uhlenbeck one.

We could have done directly the same computation with the Γ induced from the $SO(d)$ one. We obtain

$$\Gamma(\log \det M, M_{ij}) = 4(\delta_{ij} - N_{ij}).$$

With

$$\Gamma(\log\det(\mathrm{Id} - M), M_{ij}) = -4M_{ij},$$

then one gets

$$\rho = \det(M)^{(q-p-1)/2}\det(\mathrm{Id} - N)^{(d+q-1-p)/2}.$$

14. $SU(d)$ **Brownian motion**

$$\begin{cases} L(z_{ij}) = -(d^2-1)z_{ij}, \\ \Gamma(z_{ij}, z_{kl}) = z_{ij}z_{kl} - dz_{il}z_{kj}, \\ \Gamma(z_{ij}, \bar z_{kl}) = d\delta_{ik}\delta_{jl} - z_{ij}\bar z_{kl} \end{cases}$$

One may product the same kind of images that the $SO(d)$ case. There are however some subtle differences. We also would do the same for $SP(d)$.

5 Models with polynomial eigenvectors

It may happen, as it is the case for the Ornstein-Uhlenbeck operator, and also for the Laguerre and Jacobi ones, that the generator may be diagonalized in a complete system of polynomials, which are therefore orthogonal as eigenvectors of a symmetric operator with different eigenvalues.

Indeed, in dimension one, these 3 examples are the only ones which satisfy this property up to affine transformations. In higher dimension, The situation appears in many situations, for example when dealing with the sphere, or as we shall se later with compact Lie groups.

In this part, we want to address the problem of describing the diffusions, symmetric in $\mathcal{L}^2(\mu)$ which have the property that they admit a complete system of polynomial eigenvectors.

5.1 A few preliminary remarks

1. First, we shall work in some open domain $\Omega \subset \mathbb{R}^n$. A similar problem makes sense in an algebraic manifold (and we shall encounter some of them, for example the sphere), but the basic analysis is then much more complicated.
2. Then, we shall restrict the analysis to the case where μ is a probability measure (il we want polynomials to be integrable), moreover with a density with respect to the Lebesgue measure. Moreover, we shall restrict the analysis to the case where

the polynomials are dense in $\mathcal{L}^2(\mu)$. It is enough for this that the measure μ has an exponential moment (there exists $\varepsilon > 0$ such that $\int \exp(\varepsilon|x|)d\mu(x) < \infty$. Of course, it is always satisfied when Ω is bounded.

3. We want to be able to rank the polynomials with respect to some order. For this, if we are in $\mathbb{R}^d$, we chose a sequence of positive integers $(a_1, \cdots, a_d)$ and decide that the degree of $x_1^{n_1} \cdots x_d^{n_d}$ is $\sum_i a_i n_i$. Then, we denote $\mathcal{P}_n$ the space of polynomials with total degree less than n, and we have $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, $\cup_n \mathcal{P}_n = \mathcal{P}$, where $\mathcal{P}$ is te space of all polynomials. Moreover, $\mathcal{P}_n$ is a finite dimensional space.

   Then, one wants for each n to find a orthonormal basis of $\mathcal{P}_n$ which are eigenvectors for L.

   One may restrict of course to the case where all the a_i have no common factor. It could be also interesting to consider irrational degrees, but then the analysis becomes more complicated.

   It will be important in many cases not to use the usual degrees ($a_i = 1, \forall i$), although we shall mainly restrict to this case, whenever the arguments are easily generalizable to the general case.

4. There are in general many bases of orthogonal polynomials, since the dimension of $\mathcal{P}_{n+1} \ominus \mathcal{P}_n$ is larger than 1. But it many cases, there will be a unique such basis of eigenvectors of L (when the eigenvalues are simple) , so that we shall chose indeed a specific basis of orthogonal polynomials for $\mathcal{L}^2(\mu)$.

5. Since L maps $\mathcal{P}_n$ into itself, it maps $\mathcal{P}_1$ into $\mathcal{P}_{a_i}$ into itself and $\mathcal{P}_{a_i+a_j}$ also.

   Therefore, if the generator is $\sum_{ij} g^{ij}\partial^2_{ij} + \sum_i b^i \partial_i$, since $b_i = \mathrm{L}(x_i)$ and $g^{ij} = \Gamma(x_i, x_j)$, b_i are polynomials with degree at most a_i and g^{ij} are polynomials with degree at most $i + j$.

6. If we have an operator L which maps $\mathcal{P}_n$ into $\mathcal{P}_n$ for any n, and which is symmetric on $\mathcal{P}_n$ for the $\mathcal{L}^2(\mu)$ structure (which supposes in fact that the integration by parts formula is valid for a pair of polynomials), then, one may find for each n a basis of eigenvectors for L on $\mathcal{P}_n$. Then, we may construct a family of orthogonal polynomials which are eigenvectors for L.

5.2 The boundary equation

We shall restrict for simplicity to the case where Ω is bounded (this condition may be relaxed in some of the items below), with moreover a piecewise $\mathcal{C}^1$ boundary. (Indeed, the only restriction on the boundary is that the Stokes formula is valid on Ω). We shall loose no generality if we just consider piecewise $\mathcal{C}^\infty$ boundary, due to the next result.

We then consider the following data : $(\Omega, \mathrm{L}, \rho)$, where Ω is a bounded open set with the above piecewise smooth boundary, L is an elliptic diffusion operator on Ω, and ρ is the density with respect to the Lebesgue measure of the reversible probability measure for L (Recall that this amounts to describe indeed (Ω, Γ, ρ).)

If we may find some weighted degree for which the operator admits such a polynomial basis of eigenvectors, we shall call that a **polynomial diffusion model**, PDM in short.

If we restrict out attention to bounded domains $\Omega \subset \mathbb{R}^d$, with piecewise $\mathcal{C}^1$ boundary, non degenerate matrix g and finite measure, the possibility of having this situation is quite restrictive and imposes strong conditions on the domain Ω. In particular, the boundary must be an algebraic surface.

Before going into the details, let us make some precision. We shall say that $\partial\Omega$ is an algebraic set with reduced equation $\{P_1 \cdots P_k = 0\}$ if

1. Each polynomial P_i is a real polynomial, irreducible in the complex field.
2. For each regular point $x \in \partial\Omega$, there exists a neighborhood $\mathcal{V}(x)$ which contains x and a unique i such that $\mathcal{V}(x) \cap \partial\Omega = \mathcal{V}(x) \cap \{P_i = 0\}$.
3. For $i = 1 \cdots k$, there exist a regular point $x \in \partial\Omega$ such that $P_i(x) = 0$.

Then, we have the following

Theorem 5.1. *For any bounded polynomial model with a piecewize smooth boundary, one has*

1. *The boundary $\partial\Omega$ is included in an algebraic surface with reduced equation $\{P = 0\}$, where P is a polynomial which may we written as $P_1 \cdots P_k$, where the polynomials P_i are real, and complex irreducible.*
2. *The polynomial P divides $\det(g^{ij})$ (that we write $\det(\Gamma)$ in what follows, and which is a polynomial with degree at most $2\sum_i a_i$).*
3. *For each irreducible polynomial P_r appearing in the equation of the boundary, there exist polynomials $L_{i,r}$ with degree at most a_i such that*

$$\forall i = 1, \cdots, d, \ \sum_j g^{ij} \partial \log P_r = L_{i,r}. \tag{5.12}$$

4. *Let Ω be a bounded set, with boundary described by a reduced polynomial equation $\{P_1 \cdots P_k = 0\}$, such that there exist a solution $(g^{ij}, L_{i,k})$ to equation (5.12) with (g^{ij}) positive definite in Ω. Call $\Gamma(f,f) = \sum_{ij} g^{ij}\partial_i f \partial_j f$ the associated squared field operator. Then for any choice of real numbers $\{\alpha_1, \cdots, \alpha_k\}$ such that $P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ is integrable over Ω for the Lebesgue measure, setting*

$$\rho_{\alpha_1,\cdots,\alpha_k}(dx) = C_{\alpha_1,\cdots,\alpha_k} P_1^{\alpha_1} \cdots P_k^{\alpha_k} dx,$$

   where $C_{\alpha_1,\cdots,\alpha_k}$ is a normalizing constant, then $(\Omega, \Gamma, \rho_{\alpha_1,\cdots,\alpha_k})$ is a PDM.
5. *When $P = C\det(\Gamma)$, that is when those 2 polynomials have the same degree, then there are no other measures μ for which (Ω, Γ, μ) is a PDM.*

6. *The general form of the measure is the following.*
   Suppose that the determinant Δ of (g^{ij}) writes $\Delta = P_1^{m_1} \cdots P_p^{m_p}$, where P_i are real irreducible. Let J the set of indices $i \in \{1, \cdots, p\}$ such that P_i is complex reducible. Then, there exist real constants (α_i, β_j), and some polynomial Q with $\deg(Q) \le 2\sum_i a_i - \deg(\Delta)$, such that

$$\rho = \prod_i |\Delta_i|^{\alpha_i} \exp\Big(\frac{Q}{\Delta_1^{m_1-1} \cdots \Delta_p^{m_p-1}} + \sum_{j\in J} \beta_j \arctan \frac{\mathcal{I}_j}{\mathcal{R}_j}\Big). \tag{5.13}$$

Remark 5.2. *Equation (5.12), that we shall call the boundary equation (not to be confused with the equation of the boundary), may be written in a more compact form $\Gamma(x_i, \log P_r) = L_{i,r}$. Thanks to the fact that each polynomial P_r is irreducible, this is also equivalent to the fact that $\Gamma(x_i, \log P) = L_i$, for a family L_i of polynomials with degree at most 1.*

Remark 5.3. *In the non compact case, the boundary equation and the form of the measure is still valid, smooth functions with compact support are dense in the $\mathcal{L}^2$ space. Then, the boundary may not have maximal degree.*

Remark 5.4. *In the compact case, we know no examples where the factors of the determinant which do not appear in the boundary polynomial may appear in the measures.*

5.3 Examples

We shall see how many such polynomial models may be constructed from Lie group action : images of $SO(d)$, $SU(d)$, the matrix Jacobi processes , the simplex, the balls, the matrix simplex, the eigenvalues of Gaussian matrices (to be seen below in more details), etc.

5.4 Soukhanov's theorem

For the natural degree, it asserts that whenever the degree of the boundary is maximal (that is $2d$), then the metric is the product of Einstein metrics. One may not however exclude that these metrics may have negative Ricci curvature, although we only know examples with non negative Ricci. The proof is quite technical, but relies on a simple but important observation. The boundary equation may be translated in the following fact. Near a regular point of the boundary, one may consider the double cover of Ω, that is the surface with equation $Z^2 = P(X)$, where $\{P(X) = 0\}$ is the equation of the boundary. Then, the metric g may be lifted into a smooth metric on this surface, invariant under $Z \mapsto -Z$. The Ricci curvature may then be expressed as rational functions of degree 0, which may be infinite only on the set $\Delta = 0$, where Δ is the determinant of the metric. If the boundary has maximal degree, then this curvature may not be infinite on this set, and therefore it may be expressed as polynomials of degree 0, that is constants.

It remains some work to decompose the space into product of spaces of constant Ricci curvature, which may be done through the diagonalization of the *Ricci* tensor.

5.5 Consequences of the Boundary equation

1. **Identification of image measures.** We shall many examples where the image measure is of the form $CP_1^{a_1}\cdots P_k^{a_k}dx$, where $P_1\cdots P_k=0$ is the reduced equation of the boundary.
   However, in a polynomial model, whenever the boundary is not of maximal degree (or more generally when there exists some factors in $\det(\Gamma)$ which does not appear in the boundary equation), there could be more general invariant measures for a PDM, and therefore for an image measure. In the bounded case, we never observed this phenomena up to now.
2. h transforms
   When $\mathrm{L}h=\lambda h$ with $h>0$, then $\mathrm{L}^{(h)}(f)=\frac{1}{h}\mathrm{L}(fh)-\lambda f$ is a generator of a Markov process. If $h\to 0$ at the boundary of Ω, the operator $\mathrm{L}^{(h)}$ may be interpreted as a conditioning of X_t) to stay for ever in Ω.
   If L is a polynomial model, and when the measure is $\rho=P_1^{a_1}\cdots P_k^{a_k}$, We have
   $$\Gamma(x_i,\log P_k)=L_{i,k},$$
   with $L_{i,k}$ degree a_i. Then, with $c_k=\sum_i\partial_i L_{i,k}$. and with $h=1/\rho$,
   $$\mathrm{L}(h)=-(\sum_k \alpha_k c_k)h.$$
   As soon as the density ρ os infinite at the boundary of the domain (and this is the case whenever L is a Laplace operator), then we get a conditioning result. This covers all the cases mentioned above (even in many non bounded and infinite measure cases, where the boundary equation appears to be valid).
   For example, let us mention a few consequences
   (a) A real Brownian motion conditioned to never reach 0 has the law of the norm of a three dimensional Brownian motion
   (b) The same is true for a Ornstein-Uhlenbeck process
   (c) A two dimensional Brownian motion conditionned to never reach the boudaries of an equilateral triangle has the law of the spectrum of an $SU(3)$ Brownian matrix (through a natural diffeomorphism).
   (d) The spherical Brownian motion on $\mathbb{S}^d$ conditioned to never reach the an equator has the law of the projection on $\mathbb{S}^d$ of a $\mathbb{S}^{d+2}$ Brownian motion
3. Spectral measures.

6 Spectral measures, Dyson processes and principal value decompositions

We propose in this section a method which leads to simple and intrinsic computations on spectral measures in various models. The central idea is that, in order to deal with empirical measures for some finite point process in $\mathbb{R}^n$, that is the symmetric functions of some random system of points $(\lambda_1, \cdots, \lambda_n)$, it is often more convenient to use the elementary symmetric functions of those variables, in other words to use the characteristic polynomial $P(X) = \prod(X - \lambda_i)$. Now, if we are dealing with some diffusion process, we look at this polynomial (or more precisely the coefficients of this polynomial) as a process.

6.1 Polynomial with Brownian roots

This is the simplest case, where the computations are easy to deal with. It allows to perform some basic computations that may be used in a much wider context.

Start with a polynomial with Brownian roots.

$$P(X) = \prod_{i=1}^{n}(X - x_i) = \sum_{i=0}^{n} a_i X^i,$$

such that $(-1)^i a_i(x_1, \cdots, x_n)$ are the elementary symmetric functions. If we want to describe the image of the Laplace operator Δ on $\mathbb{R}^n$ under symmetric functions of $(x_1, \cdots, x_n)$, we may look at smooth functions $F(a_0, \cdots, a_{n-1})$. At least in the Weyl chamber $\{x_1 < x_2 \cdots < x_n\}$, the application $(x_1, \cdots, x_n) \mapsto \Phi(x_1, \cdots, x_n) = (a_0, \cdots, a_{n-1})$ is a local diffeomorphism. We first have to look at the image of the Lebesgue measure $dx = dx_1 \cdots dx_n$ under Φ. For this, let us introduce the discriminant.

Resultant, discriminant

For two monic polynomials $P(X) = \sum_{i=0}^{n} a_i X^i$ and $Q(X) = \sum_{i=0}^{p} b_i X^i$, the resultant $R(P, Q)$ is a polynomial in the coefficients $(a_0, \cdots, a_{n-1}, b_0, \cdots, b_{p-1})$ which vanishes exactly when P and Q have a common root (in the complex plane). Indeed, $R(P, Q)$ is the determinant of the $n \times p$ Sylvester matrix

$$\begin{pmatrix}
1 & a_{n-1} & a_{n-2} & \cdots & a_0 & 0 & \cdots & 0 \\
0 & 1 & a_{n-1} & \cdots & a_1 & a_0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & a_2 & a_1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & a_{p-2} & \cdots & a_1 & a_0 \\
1 & b_{p-1} & b_{p-2} & \cdots & b_0 & 0 & \cdots & 0 \\
0 & 1 & b_{p-1} & \cdots & b_1 & b_0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & b_2 & b_1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & b_1 & b_0
\end{pmatrix}$$

It can be viewed as the determinant of the following system of linear equations in the unknown variables $\{1, X, \cdots, X^{n+p-1}\}$:

$$\{P(X) = 0, XP(X) = 0, \cdots, X^{p-1}P(X) = 0, Q(X) = 0, XQ(X) = 0, \cdots, X^{n-1}Q(X) = 0\}.$$

It turns out that, when $P(X) = \prod_i (X - x_i)$ and $Q(X) = \prod_j (X - y_j)$, then $R(P,Q) = \prod_{i,j}(x_i - y_i)$.

The discriminant discr(P) is $D(P) = (-1)^{n(n-1)/2} R(P, P')$ and expresses a necessary and sufficient condition for P to have a double root. Then, when $P(X) = \prod(X - x_i)$, one has $\mathrm{discr}(P) = \prod_{i<j}(x_i - x_j)^2$.

Proposition 6.1. *The image measure of dx under Φ is*

$$d\mu_0 = n!|\mathrm{discr}(P)|^{-1/2} 1\!\!1_{\mathcal{D}>0} da_0 \cdots da_{n-1},$$

where D is the connected component of the set $\{\mathrm{discr}(P) > 0\}$ where all the roots of the polynomial P are real.

Proof. — By induction on the degree n.

∎

Proposition 6.2.

1. *For any $X \in \mathbb{R}$, $\Delta(P(X)) = 0$*
2. *For any $(X, Y) \in \mathbb{R}^2$,*

$$\Gamma(\log(P(X), \log P(Y)) = \frac{1}{Y - X}\Big(\frac{P'(X)}{P(X)} - \frac{P'(Y)}{P(Y)}\Big). \tag{6.14}$$

Proof. — The first assertion is immediate, since every function a_i is an harmonic function on $\mathbb{R}^n$ (as a polynomial of degree 1 in any coordinate x_i).

For the second, one has

$$\Gamma(\log P(X), \log P(Y)) = \sum_i \partial_{x_i} \log P(X) \partial_{x_i} \log P(Y) = \sum_i \frac{1}{(X - x_i)(Y - x_i)}.$$

But

$$\frac{1}{(X - x_i)(Y - x_i)} = \frac{1}{Y - X}(\frac{1}{X - x_i} - \frac{1}{Y - x_i})$$

and

$$\sum_i \frac{1}{(X - x_i)(Y - x_i)} = \frac{1}{Y - X}\Big(\frac{P'(X)}{P(X)} - \frac{P'(Y)}{P(Y)}\Big).$$

∎

Remark 6.3. *The metric structure is then characterized by*

$$\Gamma(P(X), P(Y)) = \frac{1}{Y - X}(P'(X)P(Y) - P'(Y)P(X)).$$

This metric structure reflects exactly the flat Euclidean structure of the variables (x_i).s

Corollary 6.4. *One has*

1. $\mathrm{discr}(P) = \det(\Gamma(a_i, a_j))$.
2. *For any* $i \in \{0, \cdots, n-1\}$, $\sum_j \Gamma(a_i, a_j)\partial_{a_j} \log \mathrm{discr}(P) = 2\sum_j \partial_{a_j}\Gamma(a_i, a_j)$.
3. *For any* $i \in \{0, \cdots, n-1\}$, $\sum_{i,j} X^i\Gamma(a_i, a_j)\partial_{a_j} \log \mathrm{discr}(P) = -P''(X)$.

Proposition 6.5. $\Gamma(P, \log \mathrm{discr}(P)) = -P''$.

If we replace Brownian motions by **Ornstein Uhlenbeck** *operators, then, same* Γ *and*

$\mathrm{L}_{\mathbb{OU}}(P) = -\sum_i x_i\partial_i P = \sum_i (n-i)a_iX^i = -nP(X) + XP'(X)$.

And for **Spherical Laplace operator**

$$\Gamma_{\mathbb{S}}(\log P(X), \log P(Y)) = \frac{1}{Y-X}\Big(\frac{P'(X)}{P(X)} - \frac{P'(Y)}{P(Y)}\Big) - \Big(n - X\frac{P'(X)}{P(X)}\Big)\Big(n - Y\frac{P'(Y)}{P(Y)}\Big)$$

$\Delta_{\mathbb{S}}(P(X)) = -2n(n-1)P(X) + 3(n-1)XP'(X) - X^2P''(X)$.

6.2 Spectral measures for real symmetric Brownian or *OU* matrices : the characteristic polynomial process.

Starting from the Γ and L on the entries of a matrix, one may look at the characteristic polynomial $P(X)$ which turns out to be $\det(X - M)$. From this, one sees that the coefficients of $P(X)$ are polynomials in the entries of M.

Moreover, in order to compute its action on $P(X)$, we use the following properties, which are direct consequences of Cramer's formulae. Looking at $\det(M)$, and writing (m^{-1}_{ij}) the entries of the inverse matrix $M = (m_{ij})$, one has

$$\begin{cases} \partial_{m_{ij}} \log\det(M) = m^{-1}_{ji} \\ \partial_{m_{kl}} m^{-1}_{ij} = -m^{-1}_{ik}m^{-1}_{kj} \end{cases}$$

.

For the Brownian motion on the Euclidean space of symmetric matrices (that is endowed with the scalar product $\langle M., N\rangle = \mathrm{trace}\,(MN)$), we consider $P(X) = \det(X\mathrm{Id} - M)$. We haves

1. $\Gamma(\log P(X), \log P(Y)) = \frac{1}{Y-X}\left(\frac{P'(X)}{P(X)} - \frac{P'(Y)}{P(Y)}\right)$
2. $\mathrm{L}P(X) = -\frac{1}{2}P''$.

From the previous analysis, we see that the measure is the Lebesgue measure in the da_i, $C\prod|\lambda_i - \lambda_j|$ in the Weyl Chamber.

6.3 Spectral measures for Hermitian matrices

Under the real form

$$\begin{pmatrix} M & A \\ -A & M \end{pmatrix}$$

$$\Gamma(\log P(X), \log P(Y)) = 2\mathrm{trace}\,(U(X)U(Y)) = \frac{2}{Y-X}(\frac{P'(X)}{P(X)} - \frac{P'(Y)}{P(Y)}).$$

$$\mathrm{L}(P(X)) = \frac{3}{2}\frac{P'(X)^2}{P(X)} - 2P''(X).$$

This implies that $P = P_1^2$ almost surely, and in fact

$$\Gamma(\log P_1(X), \log P_1(Y)) = \frac{1}{Y-X}\Big(\frac{P_1'(X)}{P_1(X)} - \frac{P_1'(Y)}{P_1(Y)}\Big),\ \mathrm{L}(P_1) = -P_1''. \tag{6.15}$$

Measure $\prod |\lambda_i - \lambda_j|^2$.

6.4 Symplectic matrices (symmetric on quaternions)

Real form

$$\mathcal{M} = \begin{pmatrix} M & A^1 & A^2 & A^3 \\ -A^1 & M & A^3 & -A^2 \\ -A^2 & -A^3 & M & A^1 \\ -A^3 & A^2 & -A^1 & M \end{pmatrix}$$

Same as before, and now

$$\Gamma\big(P(X), P(Y)\big) = \frac{4}{Y-X}\big(P'(X)P(Y) - P'(Y)P(X)\big),$$

$$\mathcal{L}P = \frac{9}{2}\frac{P'^2}{P} - 5P''.$$

Shows that $P = P_1^4$, and for P_1, same grad as before and $\mathcal{L}P_1 = -2P_1''$, provides measure

$$C\prod |\lambda_i - \lambda_j|^4$$

6.5 Spectral measures on Clifford algebras

(This will be detailed in the talk of M. Zani).

A general Clifford algebra may be constructed from a finite set E, for which we construct a vector space with basis ω_A, where $A \subset E$.

$E = \{1, \cdots, n\}$, $\{\omega_A, A \subset E\}$. And $\omega_A \omega_B = (A|B)\omega_{A\Delta B}$, where $A\Delta B$ denotes the symmetric difference and $(A|B) \in \{-1, 1\}$.

For Clifford algebras, the multiplication is associative and then $(A|B) = \prod_{i\in A, j\in B}(i|j)$. For the standard Clifford algebra,

$$(i|i) = -1, (i|j)(j|i) = -1 (i \neq j).$$

With this structure, one constructs real symmetric matrices, defined as block matrices

$$M = \big(M^{A\Delta B}\big)$$

with $(M^A)^t = (A|A)M^A$.

(When $\#E = 2$, this corresponds to the real form of Hermitian matrices, when $\#E = 3$, this corresponds to quaternionic matrices.)

BM on such matrices is described by

$$\Gamma(M_{ij}^A, M_{kl}^B) = \frac{1}{2}\delta_{A,B}(\delta_{ik}\delta_{jl} + (A|A)\delta_{il}\delta_{jk}), \mathcal{L}(M_{ij}^A) = 0. \tag{6.16}$$

It turns out that looking at the spectral measure process, one finds 16 different cases : this in fact reflects Bott's periodicity, and so one is able to recover this purely algebra theorem through the analysis of the process. Moreover, one also is able to describe the multiplicity of the eigenvalues of such matrices just by looking at the generator of the characteristic polynomial.

With $U(X) = (M - X\mathrm{Id})^{-1}$,

$$\frac{\mathcal{L}(P)}{P} = \Gamma(\log P) - \frac{1}{2}\Big(\sum_{A\subset E}(A|A)\Big)(\frac{P'^2}{P^2} - \frac{P''}{P}) - 2^{p-1}\sum_{C\subset E}(C|C)H(C)\big(\mathrm{trace}\, U(X)^C\big)^2,$$

where

$$H(C) = \sum_{A\subset E}(A|C)(C|A).$$

Not always a process

$$\frac{\mathcal{L}P}{P} = \begin{cases} (2^p + 2^{2m}(-1)^m)\Big(\frac{P'^2}{P^2} - \frac{P''}{P}\Big) - \frac{1}{2}\frac{P'^2}{P^2} & \text{when } p = |E| = 4m+2 \\ (2^p + 2^{2m-1}(-1)^{m+1})\Big(\frac{P'^2}{P^2} - \frac{P''}{P}\Big) - \frac{1}{2}\frac{P'^2}{P^2} & \text{when } p = |E| = 4m \end{cases}$$

Consequence $P(X) = Q(X)^a$, where Q is a polynomial, where

$$\begin{cases} a = 2^{4q}, & \text{when } p = 8q \\ a = 2^{4q+2}, & \text{when } p = 8q+2 \\ a = 2^{4q+3}, & \text{when } p = 8q+4 \\ a = 2^{4q+3}, & \text{when } p = 8q+6 \end{cases}$$

Then, we see some 8 periodicity appearing : this is Bott's periodicity

d dimension of the irreducible spaces, β exponent in the measure, α multiplicity of the roots.

$\lvert E\rvert$	structure	d	α	β
Cl(1)	$\mathbb{C}$	2	2	2
Cl(2)	$\mathbb{H}$	4	4	4
Cl(3)	$\mathbb{H}\oplus\mathbb{H}$	4	4	4
Cl(4)	$\mathbb{H}[2]$	8	8	4
Cl(5)	$\mathbb{C}[4]$	8	8	2
Cl(6)	$\mathbb{R}[8]$	8	8	1
Cl(7)	$\mathbb{R}[8]\oplus\mathbb{R}[8]$	8	8	1
Cl(8)	$\mathbb{R}[16]$	16	16	1

$$CL(p+8)=\mathbb{R}[16]\otimes CL(p).$$

There are other algebraic structures on which one may perform such computations. The most important one may be the octonion structure, since octonions play a central rôle in many parts of mathematics (exceptional Lie groups, exceptional finite groups, exceptional root systems, exceptional symmetric cones and Jordan algebras, paralelizables spheres, etc). S. Li made the computations in this case, which appear to be much more complicated, due to the fact that the algebra is not associative, and has no natural matrix representation.

6.6 Spectral measures on $SO(d)$

We may perform the same computation for the Brownian motion on $SO(d)$, that is when the generator is the Casimir operator.

$$\begin{aligned}\Gamma(P(X),P(Y)) = \quad & d\frac{XY}{1-XY}P(X)P(Y)-\frac{1}{1-XY}\Big(XP'(X)P(Y)+YP'(Y)P(X)\Big)\\ &+\frac{1}{Y-X}\Big(Y^2P'(Y)P(X)-X^2P'(X)P(Y)\Big).\end{aligned}$$

with

$$L(\log P)=(d-X\frac{P'}{P})(1-X\frac{P'}{P})+\frac{1}{1-X^2}(2X\frac{P'}{P}-d).$$

From which

$$L(P)=-d\frac{X^2}{1-X^2}+XP'(1-d+\frac{2}{1-X^2})+X^2P''.$$

Observe that this operator acts on real polynomials such that $X^dP(1/X)=(-1)^dP(X)$, which summarizes the fact that the eigenvalues lie on the unit circle and the determinant is 1. Moreover, the metric structure is the flat (Euclidean) one, with those restrictions. One has to considers two cases, the odd or even dimension.

For the even case ($d = 2d_1$), one may consider a Brownian motion $(\lambda_1, \cdots, \lambda_{d_1})$ in $\mathbb{R}^{d_1}$ and look at the polynomial

$$P(X) = \prod_{k=1}^{d_1} (X - e^{i\lambda_k})(X - e^{-i\lambda_k}).$$

Then, the carré du champ for $P(X)$ is the same as before. For the odd case, we may consider the same as before multiplied by $(X - 1)$.

Let us look now at the spectral measures. First, the spectral measure of $M \in SO(d)$: if $P(X) = \det(M - X)$, one has

$$\begin{aligned}\Gamma(P(X), P(Y)) = \quad & d\frac{XY}{1-XY}P(X)P(Y) - \frac{1}{1-XY}\Big(XP'(X)P(Y) + YP'(Y)P(X)\Big) \\ & + \frac{1}{Y-X}\Big(Y^2P'(Y)P(X) - X^2P'(X)P(Y)\Big).\end{aligned}$$

with

$$L(\log P) = (d - X\frac{P'}{P})(1 - X\frac{P'}{P}) + \frac{1}{1-X^2}(2X\frac{P'}{P} - d).$$

From which

$$L(P) = -d\frac{X^2}{1-X^2} + XP'(1 - d + \frac{2}{1-X^2}) + X^2P''.$$

Observe that this operator acts on real polynomials such that $X^d P(1/X) = (-1)^d P(X)$, which summarizes the fact that the eigenvalues lie on the unit circle and the determinant is 1.

It is far easier to work with the characteristic polynomial P than with the spectral measure itself. Indeed, we have to take in account the fact that the eigenvalues are complex numbers with modulus one, moreover conjugate to each other. Beyond this, in odd dimension, one is always 1 and there may be an even number of eigenvalues which are -1 (this happens with probability 0 indeed). So, if we want to use the spectral values are coordinates for the spectrum, we should first remove the one in odd dimension, and then chose $\lambda_i = e^{i\theta_i}$, with $0 < \theta_1 < \cdots \theta_k < \pi$ as coordinates. Then, in this Weyl chamber, one has

$$\Gamma(\lambda_i, \lambda_j) = -\delta_{ij}\lambda_i^2, \Gamma(\lambda_i, \bar{\lambda}_j) = \delta_{ij}.$$

That is the metric is the one of independent Brownian motion on the circle.

For the spectrum, one has, for this symmetric projected operator N in dimension $p \times p$, with $P = \det(N - X\mathrm{Id})$

$$
\begin{aligned}
L\log P(X) = \quad & -2d\mathrm{trace}\,(\frac{N}{N-X\mathrm{Id})}) + 2q\mathrm{trace}\,(\frac{I}{N-X\mathrm{Id}}) \\
& -2\mathrm{trace}\,(\frac{I}{N-X\mathrm{Id}})\mathrm{trace}\,(\frac{N}{N-X\mathrm{Id}}) \\
& -2\mathrm{trace}\,(\frac{N}{(N-X\mathrm{Id})^2}) + 2\mathrm{trace}\,(\frac{N^2}{(N-X\mathrm{Id})^2}) + 2\big(\mathrm{trace}\,(\frac{N}{N-X})\big)^2,
\end{aligned}
$$

while

$$\Gamma(\log P(X), \log P(Y)) = 4\mathrm{trace}\,(\frac{N}{(N-X\mathrm{Id})(N-Y\mathrm{Id})}) - 4\mathrm{trace}\,(\frac{N^2}{(N-X\mathrm{Id})(N-Y\mathrm{Id})}).$$

Now, we have

$$
\begin{cases}
\mathrm{trace}\,(\frac{I}{N-X\mathrm{Id}}) = -\frac{P'}{P} \\[2ex]
\mathrm{trace}\,(\frac{N}{N-X\mathrm{Id}}) = p - X\frac{P'}{P} \\[2ex]
\mathrm{trace}\,(\frac{N}{(N-X\mathrm{Id})^2}) = -\frac{P'}{P} + X(\frac{P'^2}{P^2} - \frac{P''}{P}) \\[2ex]
\mathrm{trace}\,(\frac{N^2}{(N-X\mathrm{Id})^2}) = p - 2X\frac{P'}{P} + X^2(\frac{P'^2}{P^2} - \frac{P''}{P}) \\[2ex]
\mathrm{trace}\,(\frac{N}{(N-X\mathrm{Id})(N-Y\mathrm{Id})}) = \frac{1}{X-Y}(Y\frac{P'}{P}(Y) - X\frac{P'}{P}(X)) \\[2ex]
\mathrm{trace}\,(\frac{N^2}{(N-X\mathrm{Id})(N-Y\mathrm{Id})}) = p + \frac{1}{X-Y}(Y^2\frac{P'}{P}(Y) - X^2\frac{P'}{P}(X))
\end{cases}
$$

When computing $L(P(X)) = P(X)\Big(L(\log(P(X)) + \Gamma(\log P(X), \log P(X))\Big)$, then some terms of the form $X\frac{P'^2}{P}$ and $X^2\frac{P'^2}{P}$ should appear. Indeed, they cancel, whatever the parameters. This is fortunate, since then we know that if $L(P(X))$ and $\Gamma(P(X), P(Y))$ have to depend only on $P(X)$ and $P(Y)$, the resulting expressions have to be polynomials.

And indeed, it is the case. We shall see on some other models examples where it is not the case and what it tells us about the structure of the underlying matrices.

We have

$$\Gamma(\log P(X), \log P(Y)) = 4\Big(-p + \frac{1}{X-Y}\big(Y(1-Y)\frac{P'(Y)}{P(Y)} - X(1-X)\frac{P'(X)}{P(X)}\big)\Big).$$

$$L(P) = 2p(p-1-d)P + 2P'(X(d+2-2p) + p - q - 1) + 2X(X-1)P''.$$

We see that expected, this vanishes on $P(1) = 0$ for $d = p+q-1$. When $d = p+q-1$, the process lives on the set where $\{P(1) = 0\}$, that is the boundary of our domain. Does the process exist when $p+q-2 < d < p+q-1$? Does it live on $\{P(1) = 0\}$? Apparently not : when $d = p+q-1-\alpha$, $L(P(1)) = -\alpha P'(1)$ on $\{P(1) = 0\}$. So that the process cannot live on this set unless it also lives on $\{P'(1) = 0\}$. It seems that for $\lambda \in (p+q-2, p+q-1)$ it happens the same phenomenon that on the spheres, that is symmetry breaking. Now, it would be good to see what happens when $d = p+q-2$. Does the measure concentrates on $\{P(1) = 0\} \cap \{P'(1) = 0\}$. And beyond ?

An idea would be to understand the conditional law of m given the spectrum of mm^*, that is it's polar decomposition.

6.7 Spectral measures on $SU(d)$.

Performing the same computation as before, but now for the Brownian motion on $SU(d)$, we get

$$\Gamma(P(X), P(Y)) = XY\Big(P'(X)P'(Y) + d\frac{P'(X)P(Y) - P'(Y)P(X)}{X-Y}\Big).$$

(Much simpler than in the orthogonal case)

$$L(P) = d(d^2 - 1 - \lambda)P + (\lambda + 2 - 2d^2)XP' + (1+d)X^2P''.$$

The carré du champ reflects the Euclidean structure for the roots. It is the same as the carré du champ of an Euclidean Brownian $(\lambda_1, \cdots, \lambda_d)$motion in n$\mathbb{R}^d$ restricted to the set $\sum_i \lambda_i = 0$, seen through the polynomial $P(X) = \prod(X - e^{i\lambda_k})$.

6.8 Principal values for Brownian matrices

We may also consider a Brownian motion on complex or real matrices, and look at it's principal values, hat is the eigenvalues s of

$$M = mm^*,\ M_{ij} = \sum_k m_{ik}\bar{m}_{jk}.$$

or

$$M = m^*m,\ M_{ij} = \sum_k \bar{m}_{ki}m_{kj}.$$

We have

$$L(M_{ij}) = 4d\delta_{ij}, \Gamma(M_{ij}, M_{kl}) = 2(\delta_{jk}M_{il} + \delta_{il}M_{kj}),$$

the rest follows from $\bar{M}_{ij} = M_{ji}$. Invariant measure $\det(M)^{d-1}dM$.

Then, $M = VN$, V unitary, N Hermitian. Writing $N = VDV^*$, where V unitary and $D = diag(x_i)$, then Now we have Γ and L for all the elements in the polar decomposition of the complex matrix m at $V = U = \mathrm{Id}$,

$$\Gamma(U_{ij}, U_{kl}) = -2\frac{x_i^2 + x_j^2}{(x_i^2 - x_j^2)^2}\delta_{il}\delta_{jk}, \quad \Gamma(U_{ij}, \overline{U}_{kl}) = 2\frac{x_i^2 + x_j^2}{(x_i^2 - x_j^2)^2}\delta_{ik}\delta_{jl},$$

$$\mathrm{L}(U_{ij}) = \mathrm{L}(\overline{U}_{ij}) = -2\sum_{k\neq i}\frac{x_i^2 + x_k^2}{(x_i^2 - x_k^2)^2}\delta_{ij},$$

$$\Gamma(x_i, x_j) = \delta_{ij}, \quad \mathrm{L}(x_i) = \frac{1}{x_i} + 4x_i\sum_{j\neq i}\frac{1}{x_i^2 - x_j^2},$$

$$\Gamma(N_{ij}, N_{kl}) = 2\frac{x_i^2 + x_j^2}{(x_i + x_j)^2}\delta_{il}\delta_{jk}, \quad \mathrm{L}(N_{ij}) = 4\sum_r \frac{x_r}{(x_i + x_r)^2}\delta_{ij},$$

$$\Gamma(V_{ij}, V_{kl}) = -\frac{4}{(x_i + x_j)^2}\delta_{il}\delta_{jk}, \quad \Gamma(V_{ij}, \overline{V}_{kl}) = \frac{4}{(x_i + x_j)^2}\delta_{ik}\delta_{jl},$$

$$\mathrm{L}(V_{ij}) = \mathrm{L}(\overline{V}_{ij}) = -4\sum_r \frac{1}{(x_i + x_r)^2}\delta_{ij},$$

$$\Gamma(V_{ij}, x_k) = 0, \quad \Gamma(U_{ij}, x_k) = 0, \quad \Gamma(V_{ij}, U_{kl}) = \frac{2}{(x_i + x_j)^2}\delta_{il}\delta_{jk}.$$

By the property of invariance under the transformation $(V, N) \to (V_0U_0VU_0^*, (U_0U)D(U_0U)^*)$, we have at arbitrary point V, U

$$\begin{aligned}
\Gamma(U_{ij}, U_{kl})(U) &= \sum U_{ip}U_{kq}\Gamma(U_{pj}, U_{ql})(\mathrm{Id}),\\
\Gamma(V_{ij}, V_{kl})(V) &= \sum (VU)_{ip}(VU)_{kq}\bar{U}_{jr}\bar{U}_{ls}\Gamma(V_{pr}, V_{qs})(\mathrm{Id}),\\
\mathrm{L}(U_{ij}) &= \sum U_{ip}\mathrm{L}(U_{pj}), \quad \mathrm{L}(\bar{U}_{ij}) = \bar{U}_{ip}\mathrm{L}(\bar{U}_{pj}),
\end{aligned}$$

other terms such as $\Gamma(V_{ij}, U_{kl})$, $\mathrm{L}(V_{ij})$ follow the same procedure. In the end, we get the conclusion in the proposition.

$SO(d)$: For an orthogonal matrix with entries m_{ij}, one gets

$$\begin{cases} L(m_{ij}) = -(d-1)m_{ij},\\ \Gamma(m_{kl}, m_{qp}) = \delta_{(kl)(qp)} - m_{kp}m_{ql}. \end{cases}$$

$SU(d)$:

$$\begin{cases} L(z_{ij}) = -(d^2-1)z_{ij},\\ \Gamma(z_{ij}, z_{kl}) = z_{ij}z_{kl} - dz_{il}z_{kj},\\ \Gamma(z_{ij}, \bar{z}_{kl}) = d\delta_{ik}\delta_{jl} - z_{ij}\bar{z}_{kl} \end{cases}$$

In those two cases, one may project on extracted $p \times q$ matrices, as we did for the sphere.

Observe in the orthogonal case that if we project on one line, we obtain the spherical brownian motion. This is no longer the case if we project the unitary brownian motion. Even projecting on one coordinate does not give quite the projection in two dimensions of any spherical brownian motion.

7 The hypergroup property and Gasper's theorem

7.1 Generalities : The Markov sequence problem (MSP)

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where a orthonormal basis $\mathcal{B}$ for $\mathcal{L}^2(P)$ is given : $\mathcal{B} = \{f_0 = 1, f_1, \cdots, f_n, \cdots\}$.

A Markov operator K is an operator mapping bounded measurable functions to bounded measurable functions, such that $K(1) = 1$ and which is positivity preserving. It is in general represented by a kernel $K(x, dy)$ of probability measures $K(f)(x) = \int f(y)K(x, dy)$.

We are looking for such Markov operators which have the basis $\mathcal{B}$ has eigenvectors : $K(f_n) = \lambda_n f_n$. It is prettily seen that that $\lambda_0 = 1$ and that $|\lambda_i| \leq 1$, for any i. Such a sequence (λ_n) is called a Markov sequence

If the series $\sum_i \lambda_i^2$ is convergent, the series $k(x, y) = \sum_i \lambda_i f_i(x) f_i(y)$ is convergent in $\mathcal{L}^2(\mu \otimes \mu)$, and the kernel $K(x, dy)$ may be represented as $k(x, y)\mu(dy)$.

The MSP ask for the description of all Markov sequences. The set of Markov sequences is compact (for the simple convergence topology on the Markov sequences), and convex. It is therefore ebnough to describe the extremal Markov sequences.

We suppose now that Ω is embedded with some topology for which the f_i are continuous functions. let $x_0 \in \Omega$.

We say that $(\Omega, \mathcal{F}, P, \mathcal{B})$ have the semigroup property at the point x_0 if, for any $x \in \Omega$, $\frac{f_n(x)}{f_n(x_0)}$ is a Markov sequence. If such is the case, then those sequences are the extremal Markov sequences and for any Markov sequnce, there exist a probaility measure ν on Ω such that, for any n,

$$\lambda_n = \int_\Omega \frac{f_n(x)}{f_n(x_0)} \nu(dx).$$

When the hypergroup property holds at some point x_0, then one may consider the 3 variables kernel

$$K(x, y, z) = \sum_n \frac{f_n(x) f_n(y) f_n(z)}{f_n(x_0)},$$

which is not necessarily convergent in $\mathcal{L}^2(\mu^{\otimes 3})$; but when it is, it is non negative. What makes sense in general is the measure $K(x,y,z)\mu(dx)$, (as a bivariate kernel) or the measure $K(x,y,z)\mu(dy)\mu(dx)$ (as a one variable kernel).

The most common example is the series $\cos(nx)$ on $(0,\pi)$, where μ is the (normalized) Lebesgue measure on $(0n\pi)$. Then, $K(f)(x) = \frac{1}{2}(f(x+y)+f(x-y))$ is a Markov kernel (here, $x+y$ and $x-y$ have to be pulled back if necessary to $(0,\pi)$ by 2π periodicity and symmetry around 0). It is a Markov operator with Markov sequence $\cos(ny)$. The hypergroup property holds at the point $x_0 = 0$. Of course, in this example, the series providing the density kernel $k(x,y)$ is not convergent.

7.2 Different aspects of the hypergroup property

1. Multiplication formulas $f_n(x)f_n(y) = \int f_n(z)K(x,y,z)\mu(dz)$. In the above example, this is the multiplication formula for the cosine function.
2. Bivariate measures on product (copules in statistics)
3. Wave equations. It is often the case that the functions f_n are eigenvectors of some operator $\mathcal{L}$ with eigenvalues μ_n. Then, a Markov density kernel $k(x,y)$ is a solution of the equation $\mathcal{L}_x k = \mathcal{L}_y k$. Then, the hypergroup property translates into the following fact.

   The solutions of the equation $\mathcal{L}_x F = \mathcal{L}_y F$, on $\Omega\times\Omega$, with boundary condition $F(.,x_0) := \delta_x$ is positive. Another reformulation is that any solution of $\mathcal{L}_x F = \mathcal{L}_y F$ starting from a non negative function at the level $x = x_0$ is non negative on $\Omega\times\Omega$. Since the operator $\mathcal{L}_x - \mathcal{L}_y$ is in general not elliptic or parabolic, this positivity preserving property is quite unexpected.

7.3 Examples

1. **Class functions on a finite group.** We consider some finite group G, and consider te functions which are invariant under conjugacy, that is $f(x) = f(g^{-1}xg)$ for any $g\in G$. They are indeed functions on the quotient space $\dot{G}$ formed with the conjugacy classes. Then, the uniform measure on G induces a natural probability measure on $\dot{G}$, where the measure of a class is proportional to it's size. A natural $\mathcal{L}^2$ basis for this space is formed by the characters of the group, that is the traces of the non equivalent irreducible representations. It turns out that, when restricting the convolution to class functions, this operation becomes commutative. Then, then operator $f\mapsto \delta_x * f$ is a markov operator, and $\delta_x * \chi = \frac{\chi(x)}{\chi(e)}\chi$ for any character, where e is the identity element. Then, the hypergroup property holds in this case with $x_0 = e$.
2. **Achour-Trimèche's result.** Another striking example of the hypergroup property is the following. Consider a symmetric interval $[-1,1]$ and a probability measure μon it, with a log-concave smooth density ρ, that we assume moreover symmetric around the origin. Consider then the operator $\mathcal{L}(f) = f'' + \frac{\rho'}{\rho}f'$, which

is symmetric in $\mathcal{L}^2(\mu)$. Assume that the spectrum of $\mathcal{L}$, with Neuman boundary conditions, is discrete (it is enough for this that $\log \rho$ is bounded on the interval), and let (f_n) be the eigenvectors. Then, $f_0 = 1$ and (f_n) is an orthonormal $\mathcal{L}^2(\mu)$ basis.

Then, Achour-Trimèche's theorem asserts that this basis has the hypergroup property, where $x_0 = \pm 1$. This result is far from trivial and we shall not expand on it here.

3. **Gasper's result.** This is perhaps the most famous result on the subject. It concerns Jacobi polynomials on $(-1, 1)$, which are form an orthonormal basis for the measure $C_{a,b}(1-x)^a(1+x)^b dx$, with $a, b > -1$. Then, provided $b \geq a$, and $a \geq -1/2$ or $a + b \geq 0$, the hypergroup property holds for this basis with $x_0 = 1$. Once again, this pproperty is quite hard to establish, but we shall provide a simple proof inspired by a recent paper of Carlen-Geronimo-Loss, and may be extended in many settings where orthogonal polynomial are concerned, provided they are eigenvectors of some diffusion operator.

7.4 The Carlen-Geronimo Loss method

In what follows we assume that we have some topological space Ω with some probability measure μ and some $\mathcal{L}^2(\mu)$ orthonormal basis (f_n), where as before $f_0 = 1$. The fundamental assumption is that there exists a symmetric operator $\mathcal{L}$ acting on $\mathcal{L}^2(\mu)$ such that for any n, f_n is an eigenvector of $\mathcal{L}$, where the associated eigenvalue is simple. We do not require indeed that $\mathcal{L}$ is effectively defined on $\mathcal{L}^2(\mu)$, but only of the dense subspace of the finite linear combinations of the vectors f_n (the algebraic span $\mathcal{F}$ of the f_n), and be symmetric on it. Moreover, and for simplicity and since this fits with the examples below, we shall assume that $\mathcal{F}$ is an algebra of bounded functions which spans the σ-algebra, and that $\mathcal{L}(f_0) = 0$ and that.

Then, assume that we have some auxiliary space Ω_1, endowed with a probability measure μ_1 and a symmetric operator $\mathcal{L}_1$ on it. We also require that $\mathcal{L}_1(\mathbf{1}) = 0$. We moreover require a few additional actors

1. A map $\pi : \Omega_1 \mapsto \Omega$, such that $\mathcal{L}_1$ maps to $\mathcal{L}$, in the sense described above. More precisely, if we denote also by π the adjoint map $\mathcal{L}^2(\mu) \mapsto \mathcal{L}^2(\mu_1)$ defined by $\pi(f)(x_1) = f(\pi(x_1))$, we once again require $\mathcal{L}_1$ to be defined and symmetric only on the functions $\pi(f_n)$, and that $f(\pi(x))$ belongs to the algebraic span of the f_n. Then, we require $\mathcal{L}_1\pi = \pi\mathcal{L}$.
2. A map $\phi = \Omega_1 \mapsto \Omega_1$, such that, with the same notations and restrictions as before, we have $\mathcal{L}_1\phi = \phi\mathcal{L}_1$.
3. Some point x_0 in Ω such that, provided Y is a random variable taking values in Ω_1 with law μ_1, the conditional law of $\pi(\phi(Y))$ given that $\pi(Y) = x_0$ is a Dirac mass at some point $x \in \Omega$.

Proposition 7.1. *Then the main result is the following : the sequence* $(\frac{f_n(x)}{f_n(x_0)})$ *is a Markov sequence.*

Proof. — (We only provide a sketch of it).

We start with a few remarks. First, since $\mathcal{L}$ is symmetric on $\mathcal{F}$, we have that $\int \mathcal{L}(f_n)d\mu = 0$, for any $n \geq 0$, and this property entirely characterizes the measure μ, since then two probability measures which share this property coincide on $\mathcal{F}$, and the monotone class theorem allows to conclude.

From this, one may conclude that the image of μ_1 under π is μ. Indeed, if we denote $\hat{\mu}$ this image measure, we have, for $f \in \mathcal{F}$

$$\int \mathcal{L} f d\hat{\mu} = \int \pi \mathcal{L} f d\mu_1 = \int \mathcal{L}_1 \pi f d\mu_1 = 0.$$

We consider the Markov operator $K(f)(z) = \mathbb{E}(f(\pi(\Phi(Y))/\pi(Y) = z)$, where Y is distributed according the the measure μ_1. It is by construction a Markov operator, and we shall see that $K(f_n) = \frac{f_n(x)}{f_n(x_0)} f_n$.

For this, for a function $f : \Omega \mapsto \mathbb{R}$, we already introduced the notation $\pi(f) = f \circ \pi : \Omega_1 \mapsto \mathbb{R}$. Similarly, for a function $g : \Omega_1 \mapsto \mathbb{R}$, denote $\Phi(g) = g \circ \Phi$. The hypotheses translate into $\mathcal{L}_1 \pi = \pi \mathcal{L}$, and $\mathcal{L}_1 \Phi = \Phi \mathcal{L}_1$.

We shall see that $K(f_n)$ is an eigenvector of $\mathcal{L}$. Indeed, denoting by $\langle f_1, f_1 \rangle = \int f_1(x) f_2(x) d\mu(x)$ and $\langle g_1, g_2 \rangle_1 = \int g_1(y) g_2(y) d\mu_1(y)$, the operator K may be characterized by the property

$$\langle K(f_1), f_2) \rangle = \langle \Phi \pi f_1, \pi f_2 \rangle_1.$$

Now, for $f, g \in \mathcal{F}$,

$$\begin{aligned} \langle K(\mathcal{L} f), g \rangle &= \langle \Phi \pi \mathcal{L} f, \pi g \rangle = \langle \mathcal{L}_1 \Phi \pi f, \pi g \rangle = \langle \Phi \pi f, \mathcal{L}_1 \pi g \rangle \\ &= \langle \Phi \pi f, \pi \mathcal{L} g \rangle = \langle K(f), \mathcal{L} g \rangle = \langle \mathcal{L} K(f), g \rangle, \end{aligned}$$

from which we conclude that $\mathcal{L} K = K \mathcal{L}$ o n $\mathcal{F}$. Then, $K(f_n)$ is also an eigenvector of $\mathcal{L}$. Since the eigenspaces of $\mathcal{L}$ are one dimensional, we may conclude that there exists some constant c_n such that $K(f_n) = c_n f_n$.

Next, we observe what happens at the point x_0. By assumption, for any continuous function, $K(f)(x_0) = f(x)$, so that $c_n = \frac{f_n(x)}{f_n(x_0)}$.

∎

To apply this result to the hypergroup property, one now needs to construct such a model Ω_1 with enough functions ϕ such that the associated points x cover all the space Ω.

7.5 Applications to some polynomial models

All the problem consists in constructing the space Ω_1, together with the maps π and Φ, and to identify x_0. For this, it is useful to have some geometric model first, for some values of the parameter from which the measure depend, and then to extend this model to the general case.

1. **Rewriting Gasper's theorem.** We recall that it concerns the Jacobi polynomials which are orthogonal with respect to the measure $Ca,b(1-x)^a(1+x)^b dx$, and eigenvector of the Jacobi operator. For simplicity, we move it on $(0,1)$ with the measure $C'_{a,b}x^a(1-x)^b dx$. We already say that when $a=(p-1)/2$ and $b=(q-1)/2$, then the Jacobi operator (up to a factor 4) is the image of the spherical Laplace operator on the unit sphere in $\mathbb{R}^{p+q}$, acting on the variable $x=x_1^2+\cdots x_p^2$. We assume here that $p\leq q$ (so that $a\leq b$). Let $X=(x_1,\cdots,x_p)$, $Y=(x_{p+1},\cdots x_{2p})$ (and for this to make sense we need $p\leq q$). Let (X,Y,Z) be the corresponding point on the sphere in $\mathbb{R}^{p+q}$.

   Then, we may chose as Ω_1 the sphere, $\mathcal{L}_1$ is the spherical Laplace operator and π is the map $(X,Y,Z)\mapsto \|X\|^2$. For the map Φ, consider any $\theta\in[0,2\pi)$, and set $\Phi_\theta(X,Y,Z)=(X_\theta,Y_\theta,Z)$, where

   $$X_\theta=\cos(\theta)X+\sin(\theta)Y, Y_\theta=-\sin(\theta)X+\cos(\theta)Y$$

   Φ_θ is a rotation in $\mathbb{R}^{p+q}$, so that it commutes with $\mathcal{L}_1$. Now, if $x_0=1$, and $x_0=\pi(X,Y,Z)$, then $Y=Z=0$, and therefore $\pi\Phi(X,Y,Z)=\cos(\theta)$. So that the conditional law of $\pi\Phi(X,Y,Z)$ knowing that $\pi(X,Y,Z)=x_0$ is a Dirac mass at $\cos(\theta)$. Those values cover all $(-1,1)$ and we have proved Gasper's theorem in this case.

   We now want to extend the result to the general case where a and b are no longer half integers. The key observation is the following : to describe the effect of the rotations Φ_θ on $x=\|X\|^2$, we only need to consider $\|X\|^2=x, \|Y\|^2=y$ together with their scalar product $u=X\cdot Y$.

   It turns out that the (x,y,u) again form a close system on the sphere. Indeed, for the spherical Laplace operator, we have

   $$\begin{aligned}&\Gamma(x,x)=4x(1-x),\Gamma(y,y)=4y(1-y),\Gamma(x,y)=-4xy;\\&\Gamma(u,u)=x+y-4u^2,\Gamma(x,u)=2u-4xu,\Gamma(y,u)=2u-4yu\end{aligned}$$

   and

   $$\mathcal{L}(x)=2p-2(p+q-1)x,\ \mathcal{L}(y)=2p-2(p+q-1)y,\ \mathcal{L}(u)=-2(p+q-1)u.$$

   This is a new polynomial model, with boundary $\{(1-x-y)(u^2-xy)=0\}$, and invariant measure

   $$C_{p,q}(1-x-y)^\alpha(xy-u^2)^\beta dxdydu,$$

with

$$\alpha = (q-p-3)/2, \ \beta = (p-3)/2.$$

Now, this new model is also valid when p and q are no longer half integers, and, for general a and b, we chose this model with invariant measure with density $C_{a,b}(1-x-y)^{b-a-3/2}(xy-u^2)^{a-1}dxdydu$, which is a probability measure as soon as $a>0$ and $b>a+1/2$ (that is this polynomial model with this carré du champ and with this reversible measure).

In this setting, the map π is the projection $(x,y,u)\mapsto x$, and the map Φ_θ writes $\Phi_\theta(x,y,u) = (x_1,y_1,u_1)$ with

$$\begin{cases} x_1 = \cos^2(\theta)x + \sin^2(\theta)y + 2\sin(\theta)\cos(\theta)u, \\ y_1 = \sin^2(\theta)x + \cos^2(\theta)y - 2\sin(\theta)\cos(\theta)u, \\ u_1 = \sin(\theta\cos(\theta)(y-x) + (\cos^2(\theta)-\sin^2(\theta)u). \end{cases}$$

which is easily seen to commute with the operator (this is obviously the case when a and b are half integers). So we get Gasper's result for the cases $b\geq a+1/2, a\geq 0$, (since the property remains true in the limit).

2. **The simplex model.** This is the most natural extension of the Jacobi polynomial case in higher dimension. Recall that the n-dimensional simplex is the domain $\Omega\subset\mathbb{R}^n$ such that $x_i>0, \sum_i x_i<1$. It is endowed with the Dirichlet measure with density

$$Cx_1^{a_1}\cdots x_n^{a_n}(1-x_1-\cdots-x_n)^{a_{n+1}}dx_1\cdots,dx_n,$$

where $a_i>-1$, $i=1,\cdots,n$.

This is also a polynomial domain, but there are many processes (that is many Γs) for which the boundary equation is satisfied. The most natural one has a simple geometric interpretation. Once again we consider a spherical Laplace operator one the unit sphere in $\mathbb{R}^N$, and chose a partition of $\{1,\cdots,N\}$ in disjoint sets $I_1,\cdots,I_{n+1}$ with $|I_j| = p_j$. Then, for a point $Y\in\mathbb{S}^{N_1}$, we set $x_j = \sum_{i\in I_j} Y_i^2$. The variables $(x_1,\cdots,x_n)$ form a closed system for the spherical Laplace operator, and the resulting image operator is an operator on the simplex with invariant measure the Dirichlet distribution, where $a_i = (p_i-1)/2$. The carré du champ on the simplex for this operator is given by $\Gamma(x_i,x_i) = 4x_i(\delta_{ij}-x_j)$.

Unfortunately, the eigenspaces for this operator is high dimensional, since the eigenvalues associated with a polynomial with highest degree term $x_1^{k_1}\cdots x_n^{k_n}$ depend only on $k_1+\cdots+k_n$. Therefore, the associated family of orthogonal polynomial is not properly defined. So we must introduce a more general polynomial operator on the simplex, with the Dirichlet distribution as reversible measure. The geometric model for this is the following. Still considering the partition $(I_1,\cdots,I_{n+1})$, one may introduce the operator

$$\mathcal{L}_{p,q} = \sum_{i\in I_p, j\in I_q}(y_i\partial_{y_j} - y_j\partial_{y_i})^2,$$

so that the spherical Laplace operator is nothing else than $\sum_{p,q\in\{1,\cdots,n+1\}}\mathcal{L}_{p,q}$.

But it turns out that the above variables $(x_1,\cdots,x_n)$ form a closed system for any $\mathcal{L}_{p,q}$, so that we may consider the operator $\sum_{p\neq q}A_{p,q}\mathcal{L}_{p,q}$ and it's image on the simplex, which is elliptic as soon as $A_{p,q}>0$ for any pair (p,q). (The action of $\mathcal{L}_{p,p}$ vanishes on the variables x_i.

One may then prove that, for a dense subset of the $A_{p,q}$ and a dense choice of the parameters a_i, the eigenvalues are simple. One may then construct as we did in the one dimensional case an intermediate model, which projects on the simplex and with proper transformations Φ. However, although the geometric picture is quite close to the one dimensional case, the construction is more complicated. If we rank the integers $p_1\leq p_2\leq\cdots\leq p_n$, we extract first from each set I_k some subset of size p_1, and consider the associated vectors $X_k\in\mathbb{R}^{p_1}$. We also consider the vector Y_k formed with the p_k-p_1 coordinates in I_k which do not appear in X_k. Then, the process on the simplex is the image of the similar process on the sphere through the map $X\mapsto(\|X_1\|^2,\|X\|_2^2+\|Y_2\|^2,\cdots,\|X_n\|^2+\|Y_n\|^2)$.

Then, the rotations Φ_θ may be replaced by the various rotations $(X_i,X_j)\mapsto\cos(\theta)X_i+\sin(\theta)X_j,-\sin(\theta)X_i+\cos(\theta)X_j)$. For the non geometric case (that is when the parameters p_i are no longer integers), one may use as intermediate model the model consisting of the variables $x_i=\|X_i\|^2$, $y_i=\|Y_i\|^2$ and $u_{ij}=X_i\cdot X_j$.

The image measure may then be expressed using the determinant of the Gramm matrix of the vectors X_i, together with the variables y_i. It becomes a bit technical, although a direct generalization of the previous one dimensional case, and we do not expand on it

3. **The Deltoid model.**

This is one example of the application of the model to some affine root system (here A_2). It concerns the action of the 2 dimensional Laplace operator on functions which are invariant under the symmetries around the lines of a triangular lattice in the plane. Those functions are indeed functions of the real and imaginary parts of the function $\mathbb{R}^2\mapsto\mathbb{C}:Z(x)=\exp(ix\cdots e_1)+\exp(ix\cdots e_2)+\exp(ix\cdots e_3)$, where e_I are the 3 third roots of unity in the complex plane. It turns out that the image $Z(\mathbb{R}^2)$ is the deltoid domain, which is one of the 11 polynomial domains in $\mathbb{R}^2$ with the usual degree. If, for 3 complex numbers satisfying $|z_i|=1$ and $z_1z_2z_3=1$, we write $Z=z_1+z_2+z_3$, and look at the discriminant $Q(Z,\bar{Z})$ of the polynomial $(X-z_1)(X-z_2)(X-z_3)=X^3-ZX+\bar{Z}X-1$, the equation of the boundary is $Q(Z,\bar{Z})=0$ and the measure may be written as $|P(Z,\bar{Z})|^\alpha dZd\bar{Z}$.

There are two geometric cases : $\alpha=-1/2$, which corresponds to the image of the 2 dimensional Laplace operator, and $\alpha=1/2$, which corresponds to the Casimir operator on $SU(3)$ acting on the trace of a matrix (which in this case collects all the spectral information and therefore form a closed system).

One again, one may construct an intermediate model for the general case (which is 6 dimensional in this case). It consists in the following observation : for the Casimir operator on $SU(3)$, the 3 diagonal entries form a closed system. Then, this system provides a domain with an associated Γ operator, for which one may adjust the measure such that the map $\pi : (z_1, z_2, z_3) \mapsto z_1 + z_2 + z_3$ projects this model on the deltoid model. The rotations are now expressed as $\Phi_\theta : (z_1, z_2, z_3) \mapsto (z_1 e^{i\theta}, z_2 e^{i\phi}, z_3 e^{-i\theta - i\phi})$.

We are in situation to prove the hypergroup property in this case. But an extra complication comes from the fact that the eigenspaces are 2 dimensional, and then the representation of Markov kernels have a most complex form. Once again, we do not expand on it.

The case of the root system A_n, or the model obtained from traces of $SU(n)$ matrices, remainss completely open.

8 Models on the boundaries

In many models, we may see that there are some parameters for which the process has a nice density as long as the parameters are beyond some limit. At the limit, the measure is singular and has support the boundary of the set. For example, on the projection of the sphere in dimension p, with $L(z_i) = -(d-1)z_i$, when $d \to p-1$, one converges to the spherical model on the sphere $\mathbb{S}^{p-1} \subset \mathbb{R}^p$. This example is simple to analyse. But it becomes more tricky for example for the projection of $SO(d)$ on $p \times q$ matrices, where the measure concentrates on $\det(\mathrm{Id} - mm^*) = 0$ when $d \to p + q - 1$. It is even worse that in this case we may suspect a lot of thresholds, when successively one, two or more eigenvalues of the matrix $\mathrm{Id} - mm^*$ are set to 0, up to the end where $p = q = d$ and all eigenvalues are 0 and the matrix is carried by the $SO(d)$ group.

We have a good way to identify the Lebesgue measure (and therefore the density with respect to it) in a system of coordinates through the fact that $\int \partial_i f d\lambda = 0$, for smooth compactly supported functions.

May we do that on the submanifold $\{P = 0\}$, around some non singular point ?

Indeed, the surface Lebesgue measure may be represented as follows. If we start from a naive representation, say that we express the last coordinate in terms of the others. Then, the surface measure is nothing else than the Riemannian measure for the surface. In coordinates $(x_1, \cdots, x_{d-1})$, one has, for the Euclidean metric in the ambiant space

$$|dx|^2 = (\delta_{ij} + u_i u_j) dx_i dx_j,$$

where $u_i = \frac{\partial_i P}{\partial_d P}$, writing $dx_d = -\sum_i^{d-1} u_i dx_i$. Therefore, in this system of coordinates, the surface measure writes $\det(\mathrm{Id} + u \otimes u)^{1/2} dx_1 \cdots dx_{d-1}$. But $\det(\mathrm{Id} + u \otimes u) = 1 + |u|^2$. Then, the surface measure in the coordinates $(x_1, \cdots, x_{d-1})$ is $\frac{|\nabla P|}{|\partial_d P|} dx_1 \cdots dx_{d-1}$. Then, one sees that, for this surface measure

$$\int V_{id}(f) d\sigma = 0,$$

where

$$V_{id} = \frac{1}{|\nabla f|}(\partial_d P \partial_i f - \partial_i P \partial_d f).$$

(Those vectors are obviously tangent to the surface $\{P = 0\}$. Observe that we could do the same for any other coordinates, such that they are linked, and in fact we just get $d-1$ independent ones, with moreover their commutators. Of course, on the sphere for example, this is just the infinitesimal rotations. This may certainly lead to the identification of the density of the associated operators with respect to the surface measure indeed.

In our models however, we consider the limit $C_\epsilon P^{-1+\epsilon} \mathbb{1}_{P>0} dx$ when $\epsilon \to 0$. This converges to some limit measure which has a density $\frac{1}{|\nabla P|}$ with respect to the surface measure. In our system of coordinates $(x_1, \cdots, x_{d-1})$, this surface measure has the density $\frac{1}{|\partial_d P|}$ with respect to the Lebesgue measure $dx_1 \cdots dx_{d-1}$.

But now, with this measure $d\sigma_P = \frac{1}{|\nabla P|} d\sigma$, one gets

$$\int W_{i,d}(f) d\sigma_P = 0,$$

where $W_{i,d} = \partial_d P \partial_i - \partial_i P \partial_d$, which are much more at hand for algebraic computations.

One main obstacle here is that one no longer has $L = \sum_{ij} g^{ij} W_i W_j + b^i W_i$, and moreover, those W_i do not commute. However, we may make use of the fact that $\Gamma(P, x_i) = a x_i P$.

Our problem now is to develop the analogy with the formula $\frac{1}{\rho}\partial_i(g^{ij}\rho\partial_j)$, where V_{id} would replace ∂_i, with the major difference that the change of variable formula is expressed with ∂_i and not V_i.

In other words, given some process with the knowledge of $\Gamma(x_i, x_j) = g^{ij}$ and $L(x_i) = b^i$, if we know that the is a function (a polynomial) which satisfies $L(P) = \Gamma(P, P) = 0$ on $\{P = 0\}$, how one would describe the invariant measure for the process living on $\{P = 0\}$ from g^{ij} and b^i ? One should now be in situation to describe from Γ and L the density of the invariant measure with respect to the surface measure, as we did for the Lebesgue measure.

Indeed, around a non singular point of the surface $\{P = 0\}$ where $\partial_d P \neq 0$, in the coordinate system $(x_1, \cdots, x_{d-1})$, one may compare the formulas for the density.

On the one hand, we have, outside $\{P = 0\}$, for the drift $b^i = L(x^i)$

$$b^i = \sum_{j=1}^d g^{ij} \partial_j \log P^{-1} + \sum_{j=1}^d \partial_j g^{ij}.$$

On the other, in this local system of coordinates

$$b^i = \sum_{j=1}^{d-1} g^{ij} \partial_j \log \frac{|\nabla P|}{\partial_d P} + \sum_{1}^{d-1} \partial_j g^{ij}.$$

Comparing, we should end up, on $\{P=0\}$, with

$$\lim_{P\to 0}\sum_{j=1}^{d} g^{ij}\partial_j \log P^{-1} + \partial_d g^{id} = \sum_{1}^{d-1} g^{ij}\partial_j \log\frac{|\nabla P|}{\partial_d P}.$$

This should rely on the sole equations, on $\{P=0\}$

$$L(P)=0, \Gamma(P,x_i)=0.$$

Observe also that n our polynomial systems, then $\lim_{P\to 0}\sum_{j=1}^{d} g^{ij}\partial_j \log P^{-1}$ make a perfect sense, since the boundary equations tells us that they are first order polynomials (and cx_i in all our cases).

The equations on P give us, on $\{P=0\}$

$$\sum_{ij=1}^{d} g^{ij}\partial_{ij}P + \sum_{i=1}^{d} b^i\partial_i P = 0, \tag{8.17}$$

and

$$\sum_{i=1}^{d} g^{ij}\partial_j P = 0. \tag{8.18}$$

Observe that the boundary equation $g^{ij}\partial_j \log P = L_i$, with L_i degree 1, together with the equation satisfied by the measure allows to take derivative in equation (8.18) to get

$$\sum_{ij=1}^{d} g^{ij}\partial_{ij}P + \sum_{i=1}^{d} b^i\partial_i P = (\sum_i \partial_i L_i)P.$$

In this case, $L(P)=0$ on the boundary and equation (8.17) is a consequence of (8.18).

Let us look at the example of spheres, where on the unit ball $\mathbb{B}_p$ we have $\Gamma(x_i,x_j)=\delta_{ij}-x_ix_j$ with $L(x_i)=-(d-1)x_i$. When $d-1\to p-1$, the the measure converges to the uniform measure on $\mathbb{S}^{p-1}$. But, for the process to live on $\mathbb{S}^{p-1}$, with this Γ and $L(x_i)=-\lambda x_i$, then it is necessary that $\lambda=p-1$ (that is $d=p-2$). Otherwise, for $R=\sum_i x_i^2$, the relation $L(R)=0$ would not hold true on $\mathbb{S}^{p-1}$, since then $L(R)=2(p-\lambda-R)$, while one still have $\Gamma(R,R)=4R(1-R)$. One has similarly a family of processes living on this $p-1$ sphere (indeed on half spheres), playing the same rôle as before. But then, one has to break the symmetry, for example choosing $L(x_p)$ such that it fits with the fact that $L(R)=0$, and then we have those processes on the sets $x_p>0$.

However, things seem to be more complicated for the matrix Jacobi processes when $d=p+q$ (d playing now the rôle of a parameter), since then one would also have a process with the same set of $p\times q$ matrices, with $d=p+q-2$, up to $d=\max(p,q)$. So that the process exists for any $d>p+q-1$, with a invariant probability measure which has a density with respect to the Lebesgue measure in the entries, while then it only exists whenever d is an integer taking values in $d \geq \max(p, q)$, and the law seems to be then supported by $\{P(1) = 0\}$, and then $\{P(1) = P'(1) = 0\}$, etc. This of course has to be checked precisely. We have exactly the same problem for Wishart matrices, that is mm^* where m is $p \times q$ with gaussian independent entries. (then the set of parameters is known to be the Jorgensen ensemble : continuous if $p \geq q$ and discrete afterwards).

9 Models in dimension 2 and 3

1. **The 11 compact 2-d models with the usual degree**

   We shall not extend on this part. the complete list of associated figures is given in the appendix.

   The special feature of the usual degree is that the full problem is then invariant under affine transformation (with weighted degrees, an affine transformation changes the degree of a polynomial). Then, the unique fact that the boundary staisfies the boundary equation is enough to completely describe all the possible models.

   This relies on the study of the local singularities of a curve for which the boundary equation is satisfied. For this, we consider this a a pure algebraic problem, that is we consider the curve in complex coordianates, and even in the complex projective space. Then, the main observation is that such a curve may not have any flex point of flat point (at least outside the infinity line), that is a point where it could be parametrized locally as $y = x^3 + o(x^3)$ or $y = x^4 + o(x^4)$. A generic algebraic curve however has many such points, and the fact that it has no such points imposes that it must have many singular points. These singular points and their numbers are computed through Plucker's formulas, and the resulting list of all possible remaining possibilities lead to the complete description of all the possible models.

   It turns out that, among those models, whenever the Laplace operator is an admissible solution, the curvature is constant. All the models that one may describe then (at least when the curvature is constant and the metric is unique) belong to 2 large families : the ones which may be constructed from affine root system in a Euclidean space, and the ones which may be constructed from finite subgroups of $O(3)$. But some of such examples escape to this list (the model associated with the G_2 root system for example), and to cover al such models, one has to consider models with weighted degrees.

   See the complete list with pictures at the end of these notes.

2. **The non compact 2-d models with usual degree**

   If we are interested in non compact domains, the boundary equation is not fully justified for non compact models. However, one may look for the non compact models which satisfy this boundary equation. It turns out that we are left with the products of one dimensional models, or the domains bounded by a parabola and a cuspidal cubic (equation $Y^2 = X^3$). As in dimension 1, those models appear as limits on bounded models. The hardest case seem to be the full $\mathbb{R}^2$ case (without boundary), where one may prove that the only admissible measures are Gaussian measures, although there are other operators that Ornstein-Uhlenbeck ones (for example, one may add $C(x\partial_y - y\partial_x)^2$ to an Ornstein-Uhlenbeck generator. On natural conjecture is that in $\mathbb{R}^n$ without boundaries, the only admissible measures are Gaussian measures.

3. **More models with weighted degrees : invariant theory.**

There are some other more subtle projections of the sphere, using discrete groups (and of course one may mix discrete and continuous groups). Here is a trick. In dimension 3, suppose that we have a discrete (finite) group of rotations with axes V_i, such the axes of the rotations are exchanged by the group. Consider the homogeneous polynomial $P(x) = \prod_i (V_i.x)$, invariant under the group action. With this polynomial $P(x)$, one may construct a new polynomial $Q(x)$ with the same homogeneity and which is harmonic, and invariant. Then, it happen quite often that the Laplace operator projects onto (XY), with $X = Q(x)$, $Y = \Gamma(X, X)$. We give some of these examples.

Writing $(x, y, z) \in \mathbb{R}^3$ and $x + iy = \tau$, consider $Y = \Re(\tau^n)$, $X = z$. Then, we get a process with $L(X) = -2X, L(Y = -n(n+1)Y$

$$\Gamma = \begin{pmatrix} (1-X^2) & -XY \\ -XY & (1-X^2)^{n-1} - Y^2 \end{pmatrix}$$

The boundary has 1 or two components according to n even or odd, and this gives rise to a family with one or two parameters.

We get new models changing X into X^2, or Y into Y^2 or both.

here is another model : $X = xyz, Y = x + y + z$, or $X = xyz, Y = x^4 + y^4 + z^4$ (corresponding to the symmetries of the cube).

Still another more tricky, corresponding to the symmetries of the tetrahedron $c = (1+\sqrt{5})/2$, and start from $X = (c^2x^2 - y^2)(c^2y^2 - z^2)(c^2z^2 - x^2)$, and $Y = \Gamma(X, X)$. We produce a very strange 2-d model with one parameter.

In any dimension, a good way to construct such models is to look at finite subgroups of $O(n)$, leading to some weighted polynomial models (but one may also construct weighted models in dimension 2 through other techniques, such as affine crystallographic groups).

The spherical Laplace operator is invariant under rotations. So if $G \subset O(3)$, then functions invariant under G are stable under $\mathcal{L}$ and Γ. If we may generate those polynomial invariants from polynomials, then one will get polynomial models.

Indeed, if P is a polynomial invariant under G, since the spherical Laplace operator commutes with rotations, so is $\mathcal{L}(P)$. Since we restrict our polynomials to spheres, it is enough to look for homogeneous invariant polynomials. The game is then to find some invariant polynomials $X_1, \cdots, X_k$ such that all invariant polynomials is a polynomial $Q(X_1, \cdots, X_k)$. Then, we shall get a closed system $(X_1, \cdots, X_k)$

for the Laplace operator, and therefore a weighted polynomial model, where the weight of the variable X_k is it's usual degree.

Bur if we want a model which satisfies the conditions of the theorem, we need to have no algebraic relations between the polynomials X_k. The next paragraph describes the structure of polynomial invariants in general.

Primary and secondary invariants . For any finite group acting linearly of a finite dimensional vector space, one may find invariant homogeneous polynomials θ_i and η_i such that each invariant homogeneous polynomial may be written as

$$P_0(\theta_1, \cdots, \theta_n) + \sum_{i=1}^{k} \eta_i P_i(\theta_1, \cdots, \theta_n),$$

where P_i are polynomials (in the variables $(\theta_1, \cdots, \theta_n)$. The polynomials θ_i are algebraically independent, their number do not depend of their choice (and is the dimension of the vector space as soon as the representation is irreducible). Moreover, each η_i satisfies some monic polynomial equation in the variables $\theta = (\theta_j)$, that is satisfies an algebraic identity of the form

$$\eta_i^{p_i} + \eta_i^{p_i-1} Q_{i,1}(\theta) + \cdots + Q_{i,p_i}(\theta) = 0,$$

where $Q_{i,k}(\theta)$ are polynomials in the variables $(\theta_1, \cdots, \theta_n)$. These algebraic relations are called syzygies (such algebras are called Cohen-MacCauley algebras). The number of the secondary invariant may depend on the choice of the θ_i and η_i.

Mollien's formula provides the dimension of the space of invariant homogeneous polynomials with degree n.

Let d_n be the dimension of the space of degree n homogeneous polynomials which are invariant under a group G, and define $F(G,t) = \sum_n d_n t^n$

$$F(G,t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(\mathrm{Id} - tg)}.$$

A theorem of Chevalley asserts that there are only primary invariants if and only if the group is generated by pseudo reflections (in our case, just reflections through hyperplanes) : that is Coxeter groups.

Since the bigger is the group, the fewer invariants it has, and since we are interested only in the case where the polynomials are algebraically independent, one may think we should only concentrate on the Coxeter groups. We shall see (at least in dimension 3), that it is not the case. It turns out that the syzygies may provide domains in higher dimensions which are also polynomial models. Up to now, we have no explanation for this fact. Moreover, in higher dimensions, the syzygies may not again be algebraically independent. There are second order syzygies, and so on. We did not explore is such higher degree syzygies may also lead to polynomial models.

4. **Construction of 2d and 3d models from symmetry groups in $\mathbb{R}^3$.**
   Three families of finite subgroups of $O(3)$ (classified by F. Klein).
   (a) Subgroups of the rotations with angle $2\pi/n$ in the plane
   (b) Subgroups of the isometry group of the tetrahedron (contains groups of the cube/octahedron)
   (c) Subgroups of the isometry group of the icosahedron/dodecahedron

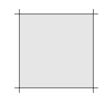
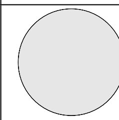
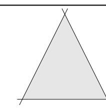
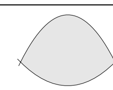
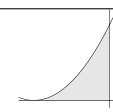
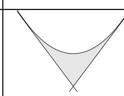
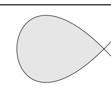
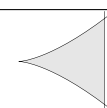
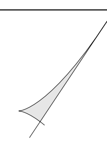
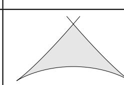
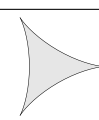
   All these are subgroups of Coxeter groups. Have to describe the polynomial invariants (primary and secondary) and the syzygies.
   Example : Plane rotation groups. . Variables (x, y, t), $z = x + iy$. Invariants $X_n = \Re(z^n)$, $Y_n = \Im(z^n)$, t, linked by the relation $X_n^2 + Y_n^2 = (1 - t^2)^n$.
   (a) Polynomial model in $2-d$ with $(X_n, z) = (X, Y)$. Domain $\{(1-Y^2)^n - X^2 \leq 0\}$: one or two irreducible factors according if n odd or even.
   (b) Polynomial model in $3 - d$ using the syzygies.
   One may write explicitly the relations between $\Gamma(U, V)$ for $U, V \in \{X_n, Y_n, t\}$ as polynomials in the variables $\{X_n, Y_n, t\}$. Then comes the surprise : the surface $X^2 + Y^2 - (1 - Z^2)^n$ is the boundary of a polynomial model in $3 - d$, where the Γ are given by the polynomial relations on the sphere.

It is worth to observe that in dimension 3, there exists a finite subgroup of $O(4)$ which is not a subgroup of a Coxeter group. For this, we are unable to associate any polynomial model with densities, and the domains (in $\mathbb{R}^4$) described by the syzygies are not polynomial domains.

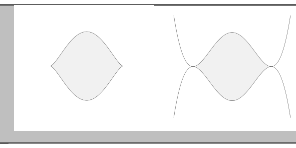
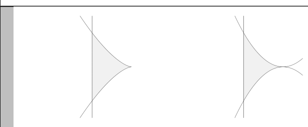
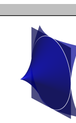
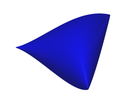
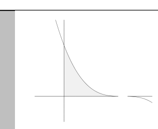
See at the end of these notes the list of such 2-d and 3-d models that one may construct in this way.

9.1 Pictures

Up to affine transformations, with the usual degree, we have 11 models in dimension 2

♯	Curv.	d(Ω)	Boundary	Picture
1	0	4	$(1-X^2)(1-Y^2)=0$	
2	$\simeq$	2	$1-X^2-Y^2=0$	
3	$\simeq$	3	$XY(1-X-Y)=0$	
4	+	$4,3$	$(1-X^2)^2-Y^2=0$	
5	+	4	$Y(1-X)(X^2-Y)$	
6	0	4	$(Y-X^2)((Y+1)^2-4X^2)=0$	
7	$\simeq$	3	$Y^2-X^2(1-X)=0$	
8	+	4	$(Y^2-X^3)(X-1)=0$	
9	+	4	$(Y^2-X^3)(2(Y-1)-3(X-1))=0$	
10	+	4	$4X^2-27X^4+16Y-128Y^2-144X^2Y+256Y^3=0$	
11	0	4	$(X^2+Y^2)^2+18(X^2+Y^2)-8X^3+24XY^2-27=0$	

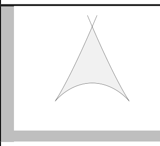
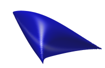
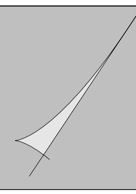
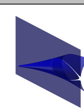
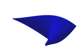
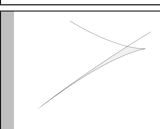
2 d and 3 d models with weighted degrees issued from 3-d rotation groups. The first series includes the models issued form the dihedral family, with $H_n(X,Y) = (1-X)^n - Y$;

Group	θ_1	θ_2	η	Ω	Boundary	Picture
$\mathcal{C}_n \vert \mathcal{D}_n$	z	X_n		$\Omega_1^{(n)}$	$H_n(X^2,Y^2) = 0$	
$\mathcal{C}_n$	z	X_n	Y_n	$\Omega_2^{(n)}$	$H_n(X^2,Y^2) - Z^2 = 0$	
$\mathcal{D}_{n,J}, \mathcal{D}_n \vert D_{2n}$	z^2	X_n		$\Omega_3^{(n)}$	$XH_n(X,Y^2) = 0$	
$\mathcal{C}_{nJ}, \mathcal{C}_n \vert \mathcal{C}_{2n}$	z^2	X_n	Y_n	$\Omega_4^{(n)}$	$X(H_n(X,Y^2) - Z^2) = 0$	
$\mathcal{D}_n$	z^2	X_n	zY_n	$\Omega_5^{(n)}$	$XH_n(X,Y^2) - Z^2 = 0$	
$\mathcal{D}_{2n,J}$	z^2	X_n^2		$D_6(\Omega_3^{(n)})$	$XYH_n(X,Y) = 0$	
$\mathcal{D}_n \vert \mathcal{D}_{2n}, \mathcal{D}_{n,J}$ $\mathcal{C}_n \vert \mathcal{C}_{2n}, \mathcal{C}_{n,J}$	z^2	X_n^2	zY_n	$\Omega_7^{(n)}$	$XH_n(X,Y) - Z^2 = 0$	
$\mathcal{C}_n \vert \mathcal{C}_{2n}, \mathcal{C}_{n,J}$	z^2	X_n^2	zX_n	$\Omega_8^{(n)}$	$(XY - Z^2)H_n(X,Y) = 0$	
$\mathcal{C}_n \vert \mathcal{C}_{2n}, \mathcal{C}_{n,J}$	z^2	X_n^2	X_nY_n	$D_9(\Omega_4^{(n)})$	$X(YH_n(X,Y) - Z^2) = 0$	

When there are two groups, the first one is for n odd, the second is for n even.

The last series provides the models issued from the the isometry groups of the cube/icosaedron/dodecahe family. Moreover, $H(X,Y) = 108X^2 - 20X - 2Y^3 + 5Y^2 - 4Y + 36XY$, and

$$\begin{aligned} S(X,Y) = \quad & 688\sqrt{5}X^4 + 6480\sqrt{5}X^3Y + 1728\,X^5 + 364\,X^3\sqrt{5} + 6042\sqrt{5}X^2Y + 23400\sqrt{5}X\,Y^2 \\ & +17050\sqrt{5}Y^3 + 1376\,X^4 + 14400\,X^3Y + 68\,X^2\sqrt{5} + 1288\,XY\sqrt{5} + 1220\sqrt{5}Y^2 \\ & +819\,X^3 + 13515\,X^2\theta_2 + 52325\,X\,\theta_2^2 + 38125\,Y^3 + 152\,X^2 + 2880\,X\,Y + 2728\,Y^2 \end{aligned}$$

G	θ_1	θ_2	η	Ω	Boundary	Picture
$\mathcal{T}\vert\mathcal{O}$	O_3	O_4		Ω_{11}	$H(X^2,Y) = 0$	
$\mathcal{T}$	O_3	O_4	O_6	Ω_{12}	$H(X^2,Y) - 4\,Z^2 = 0$	
$\mathcal{O}_J$	O_3^2	O_4		Ω_{13}	$XH(X,Y) = 0$	
$\mathcal{T}_J$	O_3^2	O_4	O_6	Ω_{14}	$X(H(X,Y) - 4Z^2) = 0$	
$\mathcal{O}$	O_3^2	O_4	O_3O_6	Ω_{15}	$Z^2 - XH(X,Y) = 0$	
$\mathcal{I}_J$	O_6	O_{10}		Ω_{21}	$S(X,Y) = 0$	
$\mathcal{I}$	O_6	O_{10}	O_{15}	Ω_{22}	$Z^2 = S(X,Y)$	

10 Bibliography

These bibliographic items have no claim to be complete. It just points to some useful books for the non specialists (or at least books that I found useful). One may find on arkiv most of the papers of the my co-authors and myself on the subject, and one may look at the references therein. Beyond this, on some specific topics :

1. On orthogonal polynomials in many variables, there are many books. The most accessible is for me Dunkl and Xu [6]. Much harder, but quite deep, I. MacDonald, [15, 16], and Geck-Jacon [10]
2. On Lie groups and algebras, there are a lot of very good books, from introductory to very technical. For me, in increasing order of accessibility Faraut [7], Gilmore [12], Knapp [14], Brocker and Dieck [3], Fulton and Harris [9], and the most complete Helgason [13]
3. On polynomial invariants, there are also some quite friendly books for non specialists : L. Smith [20] (together with his AMS survey [21]), R. Stanley [22] (more an expository paper than a book), Procesi [18, 19], and the very nice paper on invariants for the subgroups of 0(3) by Burnett Meyer [17].
4. On the analysis on spheres Stein-Weiss [23], the web page of P. Garett (http :/www.math.umn.edu/ga
5. On discriminants, beyond the classical theory quite well explained in wikipedia, one may try to look at the book of Gelfand-Kapranov-Zelevinski [11] (for an introduction to the many variables theory : quite hard to read, at least for me).
6. On the geometry of matrices Zhe-Xian Wan [25], and also Chikuse [4], but more specifically on spectral measures, see also Forrester [8]
7. On recurrence formulas, the considerations come from questions of M. Ledoux
8. On Hypergroups, there is a very large treaty Bloom and Heyer [2]. More specifically on Gelfand pairs, which is indeed at the core of the Carlen-Geronimo-Loss method, see Gerritt van Dijk [24]. And also the web page of T. Koornwinder https ://staff.fnwi.uva.nl/t.h.koornwinder/ is full of useful informations, articles and references.

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